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Limiting measures for addition modulo a prime number cellular automata

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Abstract

Linear cellular automata have many invariant measures in general. There are several studies on their rigidity: The unique invariant measure with a suitable non-degeneracy condition (such as positive entropy or mixing property for the shift map) is the uniform measure — the most natural one. This is related to study of the asymptotic randomization property: Iterates starting from a large class of initial measures converge to the uniform measure (in Cesàro sense). In this paper we consider one-dimensional linear cellular automata with neighborhood of size two, and study limiting distributions starting from a class of shift-invariant probability measures. In the two-state case, we characterize when iterates by addition modulo 2 cellular automata starting from a convex combination of strong mixing probability measures can converge. This also gives all invariant measures inside the class of those probability measures. We can obtain a similar result for iterates by addition modulo an odd prime number cellular automata starting from strong mixing probability measures.

Keywords: Linear cellular automata; stationary measures; limiting measures

1 Introduction

Let p be a prime number, and $\mathcal{A} = \{0, 1, \cdots, p-1\}$. In this paper, we consider a transformation Λ of a configuration space $\Omega := \mathcal{A}^{\mathbb{Z}} = \{\omega : \mathbb{Z} \to \mathcal{A}\}$ defined by

$$(\Lambda\omega)(x) = \omega(x-1) + \omega(x+1) \mod p$$

for $\omega \in \Omega$ and $x \in \mathbb{Z}$. This is called *addition modulo p cellular automata*. While the transformation is quite simple, the iterates exhibit various complex and interesting behaviors. For the case $p = 2$, it can be regarded as a one-dimensional version of life game ([6]; see also Exercise (2.6) of [11]). On the other hand, before that a similar kind of transformations are studied as a special case of probabilistic cellular automata in Russian literatures including [14, 15, 16] (see also [13]). Besides the delta measure concentrated on the 'all-zero' configuration, the state given by fair coin tossing, the 'most random' measure, is invariant under the transformation. After Wolfram's classification of one-dimensional "elementary" cellular automata [17], this transformation is called *rule 90*, and some of important results in [6] are independently discovered by [3].

When the distribution of the initial configuration ω is given by μ, the distribution of $\Lambda\omega$ is denoted by $\Lambda\mu$. Central problems in studying the transformation Λ are the following:

- If $\Lambda\mu = \mu$, then μ is called Λ-*invariant*: What is the Λ-invariant measures?

- What is the limiting behavior of $\Lambda^n \mu$? How about the limit in Cesàro sense: $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \Lambda^n \mu$?

Let $\boldsymbol{\theta} = (\theta_0, \theta_1, \cdots, \theta_{p-1})$ be a probability distribution on $\mathcal{A}$, namely

$$\theta_k \geqq 0 \quad \text{for } k \in \mathcal{A}, \quad \text{and} \quad \sum_{k\in\mathcal{A}} \theta_k = 1.$$

Let $(\omega(x) : x \in \mathbb{Z})$ be a doubly infinite sequence of independent, identically distributed random variables satisfying

$$\omega(x) = k \text{ with probability } \theta_k \quad [k \in \mathcal{A}].$$

The distribution of $(\omega(x) : x \in \mathbb{Z})$ is denoted by $\mu_{\boldsymbol{\theta}}$, and called a *product measure with density* $\boldsymbol{\theta}$. In particular, $\mu_{\boldsymbol{\theta}}$ with

$$\theta_k = \frac{1}{p} \quad [k \in \mathcal{A}]$$

is called the *uniform measure* and is denoted by $\mu_{1/p}$. For $a \in \mathcal{A}$, the constant configuration $\cdots aaa \cdots$ is denoted by $\boldsymbol{a}$, and the delta measure concentrated on $\boldsymbol{a}$, namely $\mu_{\boldsymbol{\theta}}$ with

$$\theta_k = \begin{cases} 1 & (k = a), \\ 0 & (k \neq a) \end{cases}$$

is denoted by $\delta_{\boldsymbol{a}}$. When $p = 2$, the product measure with $\theta_1 = \rho$ and $\theta_0 = 1 - \rho$ is called the *Bernoulli measure* with density ρ, and is denoted by β_ρ.

Suppose that $p = 2$ for the moment. Starting from a single 1, rule 90 generates Pascal's triangle mod 2 a.k.a. the pre-Sierpiński gasket, which reflects the scaling relation (see Lemma 3.5 below)

$$(\Lambda^{2^m}\omega)(x) = \omega(x - 2^m) + \omega(x + 2^m) \mod 2 \quad [\omega \in \Omega,\, x \in \mathbb{Z};\, m = 0, 1, 2, \cdots].$$

From this, we have

$$\beta_\rho\big((\Lambda^{2^m}\omega)(x) = 1\big) = \beta_\rho\big(\omega(x - 2^m) + \omega(x + 2^m) = 1\big) = 2\rho(1-\rho) \quad [m = 0, 1, 2, \cdots].$$

On the other hand, since

$$(\Lambda^{2^m-1}\omega)(x) = \sum_{j=x\pm 2^{m-1}\pm 2^{m-2}\pm\cdots\pm 2^0} \omega(j) \mod 2 \quad [\omega \in \Omega,\, x \in \mathbb{Z};\, m = 0, 1, 2, \cdots],$$

we can see that

$$\lim_{m\to\infty} \beta_\rho\big((\Lambda^{2^m-1}\omega)(x) = 1\big) = \frac{1}{2} \quad \text{if } 0 < \rho < 1.$$

Miyamoto [6] and Lind [3] proved that

- $\lim_{n\to\infty} \Lambda^n \beta_\rho$ exists if and only if $\rho \in \{0, 1/2, 1\}$. β_ρ is Λ-invariant if and only if $\rho \in \{0, 1/2\}$.
- If $0 < \rho < 1$, then the Cesàro mean $\frac{1}{N} \sum_{n=0}^{N-1} \Lambda^n \beta_\rho$ converges to $\beta_{1/2}$ as $N \to \infty$.

Cai and Luo [1] extended the above result to p odd prime:

- $\lim_{n\to\infty} \Lambda^n \mu_{\boldsymbol{\theta}}$ exists if and only if $\mu_{\boldsymbol{\theta}} = \delta_{\boldsymbol{0}}$ or $\mu_{1/p}$. Thus those are only Λ-invariant measures in the class of product measures.
- If $\theta_k < 1$ for all k, then the Cesàro mean $\frac{1}{N} \sum_{n=0}^{N-1} \Lambda^n \mu_{\boldsymbol{\theta}}$ converges to $\mu_{1/p}$ as $N \to \infty$.

The above results show that

- The uniform measure is the only invariant measure among non-trivial product measures.
- Iterates starting from non-trivial product measures converge to the uniform measure in Cesàro sense.

The former property is a kind of *rigidity* — under some condition excluding “degenerated” measures (e.g. probability measures generated from periodic points), the only invariant measure is the uniform one. Observations towards such a property can be found already in [16]. See [4] for a recent survey and related references. The latter property is called *asymptotic randomization* — convergence to the uniform measure for a large class of initial measures. Among several attempts to extend the class of measures randomized by cellular automata, Pivato and Yassawi [9] introduced *harmonically mixing measures*, and proved those measures they are randomized by non-trivial affine cellular automata. Their theory can be applied to very general settings.

This paper is an extended version of [12]. We give some rigidity results for probability measures with a mixing property with respect to the spatial shift; a general class including product measures. The rest of the paper is organized as follows: In section 2 we explain our setting and state our results. Preliminary facts are presented in section 3. Proofs of our results are given in sections 4 and 5.

2 Setting and results

We introduce several notions from ergodic theory, which are generalizations of properties of product measures $\mu_{\boldsymbol{\theta}}$.

2.1 Borel probability measures on the configuration space

For a positive integer L, let

$$\Omega_L = \mathcal{A}^L := \{\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_L) : \sigma_1, \sigma_2, \cdots, \sigma_L \in \mathcal{A}\}.$$

For $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_L) \in \Omega_L$ and $a \in \mathbb{Z}$, put

$$[\boldsymbol{\sigma}]_{a+1}^{a+L} = [\sigma_1\sigma_2\cdots\sigma_L]_{a+1}^{a+L} := \{\omega \in \Omega : \omega(a+x) = \sigma_x \, (x = 1, 2, \cdots, L)\}.$$

Such a subset of Ω is called a *cylinder set*.

The σ-algebra of events generated by all cylinder sets is denoted by $\mathcal{B}$. Hereafter we treat probability measures on $(\Omega, \mathcal{B})$, which are called *Borel probability measures* on Ω: They are uniquely determined by the probability of cylinder sets. For example, a product measure $\mu_{\boldsymbol{\theta}}$ with density $\boldsymbol{\theta} = (\theta_0, \theta_1, \cdots, \theta_{p-1})$ is characterized by

$$\mu_{\boldsymbol{\theta}}([\boldsymbol{\sigma}]_{a+1}^{a+L}) = \prod_{x=1}^{L} \theta_{\sigma_x} \quad [\boldsymbol{\sigma} \in \Omega_L] \tag{1}$$

for any $a \in \mathbb{Z}$.

For probability measures μ_1 and μ_2, we write $\mu_1 = \mu_2$ if $\mu_1(A) = \mu_2(A)$ for each cylinder set A. For a sequence of probability measures $\{\mu_n\}$ and a probability measure μ, we write $\lim_{n\to\infty} \mu_n = \mu$ if $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for each cylinder set A.

2.2 Shift-invariant measures

By (1), the probability $\mu_{\boldsymbol{\theta}}([\boldsymbol{\sigma}]_{a+1}^{a+L})$ is independent of a: This property is called *shift-invariance* of $\mu_{\boldsymbol{\theta}}$. More precisely, we define a left shift transformation T of Ω by

$$(T\omega)(x) = \omega(x+1) \quad [x \in \mathbb{Z}].$$

The inverse transformation T^{-1} is the right shift transformation. The n-fold iteration of T (resp. T^{-1}) is denoted by T^n (resp. T^{-n}). For an event $A \in \mathcal{B}$, let

$$T^{-n}A := \{\omega \in \Omega : T^n\omega \in A\} = \{T^{-n}\omega : \omega \in A\}.$$

For example,

$$\begin{aligned} T^{-n}[\boldsymbol{\sigma}]_1^L &= \{\omega \in \Omega : (T^n\omega)(x) = \sigma_x \, (x = 1, 2, \cdots, L)\} \\ &= \{\omega \in \Omega : \omega(x+n) = \sigma_x \, (x = 1, 2, \cdots, L)\} = [\boldsymbol{\sigma}]_{1+n}^{L+n}, \end{aligned}$$

where $\boldsymbol{\sigma} \in \Omega_L$. A probability measure μ on Ω is called *shift-invariant* if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$. This is equivalent to the following: For any L, n, and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_L) \in \Omega_L$,

$$\mu\left([\boldsymbol{\sigma}]_{1+n}^{L+n}\right) = \mu\left([\boldsymbol{\sigma}]_1^L\right).$$

This common value is often denoted by $\mu(\boldsymbol{\sigma})$.

2.3 Mixing properties

The following property of $\mu_{\boldsymbol{\theta}}$ is called *pairwise independence*: Let $a, b \in \mathbb{Z}$ and L, L' be positive integers. If $[a+1, a+L] \cap [b+1, b+L'] = \emptyset$, then

$$\mu_{\boldsymbol{\theta}}([\boldsymbol{\sigma}]_{a+1}^{a+L} \cap [\boldsymbol{\sigma}']_{b+1}^{b+L'}) = \mu_{\boldsymbol{\theta}}([\boldsymbol{\sigma}]_{a+1}^{a+L})\mu_{\boldsymbol{\theta}}([\boldsymbol{\sigma}']_{b+1}^{b+L'})$$

for any $\boldsymbol{\sigma} \in \Omega_L$ and $\boldsymbol{\sigma}' \in \Omega_{L'}$. In fact $\mu_{\boldsymbol{\theta}}$ has a much stronger property called *independence*, which asserts that $\mu_{\boldsymbol{\theta}}$-probability of an intersection of finitely many cylinder events with disjoint supports is given by the product of $\mu_{\boldsymbol{\theta}}$-probability of each cylinder set.

Let μ be a shift-invariant probability measure. Notions of asymptotic independence with respect to shift transformations are called *mixing properties*. (See e.g. Chapter VII of [10].) μ is called *strong mixing* if

$$\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B)$$

for all $A, B \in \mathcal{B}$. This is equivalent to the following: Let L, L' be positive integers. For any $\boldsymbol{\sigma} \in \Omega_L$ and $\boldsymbol{\sigma}' \in \Omega_{L'}$,

$$\mu\left([\boldsymbol{\sigma}]_1^L \cap [\boldsymbol{\sigma}']_{1+n}^{L'+n}\right) \to \mu\left([\boldsymbol{\sigma}]_1^L\right)\mu\left([\boldsymbol{\sigma}']_1^{L'}\right)$$

as $n \to \infty$. More generally, μ is called *r-fold mixing* if for $A, B_1, \cdots, B_r \in \mathcal{B}$,

$$\mu(A \cap T^{-n_1}B_1 \cap \cdots \cap T^{-n_r}B_r) \to \mu(A)\mu(B_1)\cdots\mu(B_r)$$

as $n_1 \to \infty$, $n_2 - n_1 \to \infty$, $\cdots$, $n_r - n_{r-1} \to \infty$. Let $\mathcal{M}_r$ be the set of r-fold mixing probability measures on Ω ($\mathcal{M}_1$ is the set of strong mixing probability measures). Note that

$$\mathcal{M}_1 \supset \mathcal{M}_2 \supset \cdots \supset \mathcal{M}_r \supset \cdots.$$

μ is called *K-mixing* if for any $A \in \mathcal{B}$,

$$\lim_{n\to\infty} \sup_{B \in \mathcal{G}_n} |\mu(A \cap B) - \mu(A)\mu(B)| = 0,$$

where $\mathcal{G}_n$ is the σ-algebra generated by $\{\omega(i) : i \geqq n\}$. It is known that μ is K-mixing if and only if the σ-algebra

$$\mathcal{G}_\infty := \bigcap_{n=1}^{\infty} \mathcal{G}_n$$

is trivial with respect to μ. Let $\mathcal{M}$ be the set of K-mixing probability measures on Ω. It is also known that

$$\mathcal{M} \subset \bigcap_{r=1}^{\infty} \mathcal{M}_r.$$

By the Kolmogorov 0-1 law, product measures $\mu_{\boldsymbol{\theta}}$ are K-mixing: Thus they have all mixing properties explained above.

2.4 Results

For a set of probability measures $\mathcal{P}$, the *convex hull* of $\mathcal{P}$ is defined by

$$\mathrm{Conv}(\mathcal{P}) := \left\{ \int_{\mathcal{P}} \mu \, d\pi(\mu) : \pi \text{ is a probability measure on } \mathcal{P} \right\}.$$

We obtain the following result for $p = 2$, which is an improvement of Theorem II.2 in [12].

Theorem 2.1. Let $p = 2$. Assume that $P \in \mathrm{Conv}(\mathcal{M}_1)$. Then $\Lambda^n P$ converges as $n \to \infty$ if and only if

$$P = \alpha\beta_0 + \alpha'\beta_{1/2} + \alpha''\beta_1$$

for some $\alpha, \alpha', \alpha'' \geqq 0$ with $\alpha + \alpha' + \alpha'' = 1$. Thus $P \in \mathrm{Conv}(\mathcal{M}_1)$ is a stationary measure for Λ if and only if

$$P = \alpha\beta_0 + \alpha'\beta_{1/2}$$

for some $\alpha, \alpha' \geqq 0$ with $\alpha + \alpha' = 1$.

Our second result is for $p > 2$, which is an improvement of Theorem II.1 in [12].

Theorem 2.2. Let p be an odd prime number, and μ be a shift-invariant, strong mixing probability measure on $\{0, 1, \cdots, p-1\}^{\mathbb{Z}}$.

$$\lim_{n \to \infty} \Lambda^n \mu$$

exists if and only if $\mu = \delta_{\mathbf{0}}$ or $\mu_{1/p}$. In particular, shift-invariant, strong mixing, Λ-invariant probability measures are only those two.

Miyamoto [7] obtained an analogous result to Theorem 2.1 for $P \in \mathrm{Conv}(\mathcal{M})$. (In fact, the proof given in [7] works for $P \in \mathrm{Conv}(\mathcal{M}_3)$.) Theorem 2.2 implies a result obtained by Marcovici (Proposition 5.5 of [5] and Proposition 3.2.2 of [2]): If $\mu \in \mathcal{M}_1$ has full support on Ω (i.e. positive probability on each cylinder set) and Λ-invariant, then $\mu = \mu_{1/p}$. In the paper of Pivato [8], much more general linear cellular automata are treated, and invariant measures in some classes of shift-mixing probability measures are investigated.

3 Preliminaries

3.1 Fourier transform

Let μ be a probability measure on $\Omega = \mathcal{A}^{\mathbb{Z}}$. For a configuration $\xi \in \Omega$, we put

$$\#\xi := \text{the number of } x \in \mathbb{Z} \text{ with } \xi(x) \neq 0.$$

Let $\Xi := \{\xi \in \Omega : \#\xi < +\infty\}$. For $\xi \in \Xi$ and $\omega \in \Omega$, we define

$$\langle \xi, \omega \rangle := \sum_{x \in \mathbb{Z}} \xi(x)\omega(x),$$

and

$$\widehat{\mu}(\xi) = F(\mu)(\xi) := \int_{\Omega} \exp\left(\frac{2\pi i}{p}\langle \xi, \omega \rangle\right) \mu(d\omega).$$

Note that $|\widehat{\mu}(\xi)| \leqq 1$ and $\widehat{\mu}(\mathbf{0}) = 1$.

The following theorem is well-known: An elementary proof is found in [12].

Theorem 3.1 (the Fourier inversion formula)**.** Let $x_0, x_1 \in \mathbb{Z}$ with $x_0 \leqq x_1$, and

$$\Xi_{x_0,x_1} := \{\xi \in \Omega : \xi(x) = 0 \text{ if } x < x_0 \text{ or } x > x_1\} \subset \Xi.$$

For any $(\sigma_{x_0}, \cdots, \sigma_{x_1}) \in \Omega_{x_1-x_0+1}$,

$$\mu([\sigma_{x_0} \cdots \sigma_{x_1}]_{x_0}^{x_1}) = \frac{1}{p^{x_1-x_0+1}} \sum_{\xi \in \Xi_{x_0,x_1}} \exp\left(-\frac{2\pi i}{p} \sum_{x=x_0}^{x_1} \xi(x)\sigma_x\right) \widehat{\mu}(\xi).$$

As its corollaries,

- For probability measures μ_1 and μ_2 on Ω, if $\widehat{\mu_1}(\xi) = \widehat{\mu_2}(\xi)$ for any $\xi \in \Xi$, then $\mu_1 = \mu_2$.
- For a sequence $\{\mu_n\}$ of probability measures on Ω and a probability measure μ on Ω, if $\lim_{n\to\infty} \widehat{\mu_n}(\xi) = \widehat{\mu}(\xi)$ for any $\xi \in \Xi$, then $\lim_{n\to\infty} \mu_n = \mu$.

For example, let us calculate the Fourier transform of the product measure $\mu_{\boldsymbol{\theta}}$: Clearly $\widehat{\mu_{\boldsymbol{\theta}}}(\mathbf{0}) = 1$. For any $\xi \in \Xi \setminus \{\mathbf{0}\}$, we have

$$\begin{aligned} \widehat{\mu_{\boldsymbol{\theta}}}(\xi) &= \int_\Omega \exp\left(\frac{2\pi i}{p}\langle \xi, \omega\rangle\right) \mu_{\boldsymbol{\theta}}(d\omega) \\ &= \prod_{x \in \operatorname{supp}\xi} \int_\Omega \exp\left(\frac{2\pi i}{p}\xi(x)\omega(x)\right) \mu_{\boldsymbol{\theta}}(d\omega) = \prod_{x \in \operatorname{supp}\xi} \left\{\sum_{k\in\mathcal{A}} \theta_k \exp\left(\frac{2\pi i}{p}\xi(x)k\right)\right\}, \end{aligned} \tag{2}$$

where $\operatorname{supp}\xi := \{x \in \mathbb{Z} : \xi(x) \neq 0\}$. In particular, the uniform measure $\mu_{1/p}$ satisfies that $\widehat{\mu_{1/p}}(\xi) = 0$ for any $\xi \in \Xi \setminus \{\mathbf{0}\}$.

3.2 Fourier transform and addition modulo p

We recall a 'duality' for addition modulo p (see e.g. section 2 of [1]).

Lemma 3.2. Let $\xi \in \Xi$ and $\omega \in \Omega$. For any $n = 1, 2, \cdots$, $\langle \xi, \Lambda^n\omega\rangle = \langle \Lambda^n\xi, \omega\rangle$.

Proof. For the case $n = 1$,

$$\begin{aligned} \langle \xi, \Lambda\omega\rangle = \sum_{x\in\mathbb{Z}} \xi(x)\cdot(\Lambda\omega)(x) &= \sum_{x\in\mathbb{Z}} \xi(x)\cdot(\omega(x-1)+\omega(x+1)) \\ &= \sum_{x\in\mathbb{Z}} (\xi(x+1)+\xi(x-1))\cdot\omega(x) = \sum_{x\in\mathbb{Z}} (\Lambda\xi)(x)\cdot\omega(x) = \langle \Lambda\xi, \omega\rangle. \end{aligned}$$

Noting that $\#(\Lambda\xi) < +\infty$ if $\#\xi < +\infty$, we can show the lemma for the general n by induction. $\square$

Lemma 3.3. $\widehat{\Lambda^n\mu}(\xi) = \widehat{\mu}(\Lambda^n\xi)$ for any $\xi \in \Xi$.

Proof. By Lemma 3.2,

$$\widehat{\Lambda^n\mu}(\xi) = \int_\Omega \exp\left(\frac{2\pi i}{p}\langle \xi, \Lambda^n\omega\rangle\right) \mu(d\omega) = \int_\Omega \exp\left(\frac{2\pi i}{p}\langle \Lambda^n\xi, \omega\rangle\right) \mu(d\omega) = \widehat{\mu}(\Lambda^n\xi).$$

$\square$

Besides the obvious Λ-invariant measure $\delta_{\mathbf{0}}$,

Lemma 3.4. The uniform measure $\mu_{1/p}$ is Λ-invariant.

Proof. For $\xi \in \Xi \setminus \{\mathbf{0}\}$, since $0 < \#(\Lambda\xi) < \infty$, Lemma 3.3 implies that

$$\widehat{\Lambda\mu_{1/p}}(\xi) = \widehat{\mu_{1/p}}(\Lambda\xi) = 0 = \widehat{\mu_{1/p}}(\xi).$$

$\square$

3.3 Strong mixing measures and addition modulo p

The following scaling relation is well-known, and found in section 4 of [1] among others.

Lemma 3.5. For any $m = 0, 1, 2, \cdots$,

$$(\Lambda^{p^m}\omega)(x) = \omega(x - p^m) + \omega(x + p^m) \mod p$$

for $\omega \in \Omega$ and $x \in \mathbb{Z}$.

Proof. Since p is prime, we can see that $(1 + x)^{p^m} = 1 + x^{p^m}$ as $\{0, 1, \cdots, p-1\}$-polynomials, from which the conclusion follows. □

We use an important formula in the proof of Theorem 1 of [7]:

Lemma 3.6. If $\mu \in \mathcal{M}_1$, then

$$\lim_{m\to\infty} F(\Lambda^{p^m}\mu)(\xi) = \widehat{\mu}(\xi)^2 \quad \text{for any } \xi \in \Xi.$$

Proof. We may assume that $\xi \in \Xi \setminus \{\mathbf{0}\}$. Lemma 3.5 implies that for any m,

$$\begin{aligned} F(\Lambda^{p^m}\mu)(\xi) &= \int_\Omega \exp\left(\frac{2\pi i}{p}\langle \xi, \Lambda^{p^m}\omega\rangle\right)\mu(d\omega) \\ &= \int_\Omega \exp\left(\frac{2\pi i}{p}\sum_{x\in \operatorname{supp}\xi} \xi(x)\cdot(\Lambda^{p^m}\omega)(x)\right)\mu(d\omega) \\ &= \int_\Omega \exp\left(\frac{2\pi i}{p}\sum_{x\in \operatorname{supp}\xi} \xi(x)\omega(x-p^m)\right)\cdot \exp\left(\frac{2\pi i}{p}\sum_{x\in \operatorname{supp}\xi} \xi(x)\omega(x+p^m)\right)\mu(d\omega). \end{aligned}$$

Letting $m \to \infty$, we obtain the conclusion by the strong mixing property of μ. □

4 Limiting measures for addition modulo p

In this section, we prove Theorem 2.2. First we prepare a simple lemma.

Lemma 4.1. Let $n > 1$ be an integer, and $\{\theta_k\}_{k\in\{0,1,\cdots,n-1\}}$ be a probability distribution on $\{0, 1, \cdots, n-1\}$. A necessary and sufficient condition for

$$\left|\sum_{k=0}^{n-1} \theta_k \exp\left(\frac{2\pi i}{n}\cdot k\right)\right| = 1$$

is $\theta_k = 1$ for some $k \in \{0, 1, \cdots, n-1\}$.

Proof. Sufficiency is obvious. To show necessity, recall that for two complex numbers z and w, $|z + w| < |z| + |w|$ if and only if $zw \neq 0$ and $\arg z \neq \arg w$. If $\theta_\ell, \theta_m > 0$ for $\ell, m \in \{0, 1, \cdots, n-1\}$ with $\ell \neq m$, then we have

$$1 = \left|\sum_{k=0}^{n-1} \theta_k \exp\left(\frac{2\pi i}{n}\cdot k\right)\right| < \sum_{k=0}^{n-1} \theta_k \left|\exp\left(\frac{2\pi i}{n}\cdot k\right)\right| = \sum_{k=0}^{n-1} \theta_k = 1,$$

a contradiction. □

Let p be a prime number, and $0^L := (0, 0, \cdots, 0) \in \Omega_L = \mathcal{A}^L$.

Lemma 4.2. Suppose that $\mu \in \mathcal{M}_1$ and $\mu(0^L) > 0$ for any L. If $\widehat{\mu}(\xi) = 1$ for some $\xi \in \Xi \setminus \{\mathbf{0}\}$, then $\mu = \delta_{\mathbf{0}}$.

Proof. Suppose that $\widehat{\mu}(\xi) = 1$ for $\xi \in \Xi$ with

$$\xi(x) = \begin{cases} a & (x = x_0), \\ 0 & (x \neq x_0) \end{cases} \quad [a \neq 0].$$

Since

$$\widehat{\mu}(\xi) = \sum_{k=0}^{p-1} \mu\left(a\omega(x_0) = k\right) \cdot \exp\left(\frac{2\pi i}{p} \cdot k\right) = 1,$$

Lemma 4.1 implies that $\mu\left(\omega(x_0) = 0\right) = \mu\left(a\omega(x_0) = 0\right) = 1$. By the shift-invariance, $\mu = \delta_{\mathbf{0}}$.

Now we turn to the case $\widehat{\mu}(\xi) = 1$ for some $\xi \in \Xi$ with $\#\xi > 1$. By the shift-invariance of μ, we can assume that there is a positive integer L such that

$$\xi(x) = 0 \text{ if } x < 0 \text{ or } x > L, \text{ and } \xi(0), \xi(L) \neq 0.$$

Since

$$\widehat{\mu}(\xi) = \sum_{k=0}^{p-1} \mu\left(\sum_{x=0}^{L} \xi(x)\omega(x) = k\right) \cdot \exp\left(\frac{2\pi i}{p} \cdot k\right) = 1,$$

Lemma 4.1 implies that

$$\mu\left(\sum_{x=0}^{L} \xi(x)\omega(x) = 0\right) = \mu\left(\xi(0)\omega(0) + \sum_{x=1}^{L} \xi(x)\omega(x) = 0\right) = 1.$$

Using the shift-invariance of μ, we can see that

$$\mu\left(\xi(0)\omega(n) + \sum_{x=1}^{L} \xi(x)\omega(x+n) = 0 \text{ for } n = 0, 1, 2, \cdots\right) = 1.$$

On the event in the left hand side, if $\omega(1+n) = \omega(2+n) = \cdots = \omega(L+n) = 0$ for some n, then $\omega(0) = 0$. This means that

$$\mu(\{\omega(0) \neq 0\} \cap [0^L]_{1+n}^{L+n}) = 0 \quad \text{for any } n.$$

By the strong mixing property of μ, letting $n \to \infty$,

$$\mu(\omega(0) \neq 0) \cdot \mu(0^L) = 0.$$

Since $\mu(0^L) > 0$, we have

$$\mu(\omega(0) \neq 0) = 0, \quad \text{i.e. } \mu = \delta_{\mathbf{0}}.$$

This completes the proof. $\square$

Now we prove Theorem 2.2. Let p be an odd prime number. We assume that μ is strong mixing and $\mu_\infty := \lim_{n\to\infty} \Lambda^n \mu$ exists. Noting that

$$\mu_\infty = \lim_{m\to\infty} \Lambda^{p^m} \mu,$$

we obtain

$$\widehat{\mu_\infty}(\xi) = \lim_{m\to\infty} \widehat{\mu}(\Lambda^{p^m} \xi) = \widehat{\mu}(\xi)^2$$

by Lemmata 3.3 and 3.6. On the other hand, since the limiting probability measure μ_∞ is Λ-invariant,

$$\widehat{\mu_\infty}(\xi) = F(\Lambda^n \mu_\infty)(\xi) = \widehat{\mu_\infty}(\Lambda^n \xi) = \widehat{\mu}(\Lambda^n \xi)^2$$

for any n. Substituting $n = p^m$ and letting $m \to \infty$, we have

$$\widehat{\mu_\infty}(\xi) = \{\widehat{\mu}(\xi)^2\}^2 = \widehat{\mu}(\xi)^4,$$

again by Lemma 3.6. Thus we have $\widehat{\mu}(\xi)^2 = \widehat{\mu}(\xi)^4$. Since p is odd, Lemma 4.1 shows that $\widehat{\mu}(\xi) = 0$ or 1. Noting that $\widehat{\mu}(\mathbf{0}) = 1$, Theorem 3.1 implies

$$\mu([0^L]_1^L) = \frac{1}{p^L} \sum_{\xi \in \Xi_{1,L}} \widehat{\mu}(\xi) = \frac{\#\{\xi \in \Xi_{1,L} : \widehat{\mu}(\xi) = 1\}}{p^L} \geqq \frac{1}{p^L} > 0 \quad \text{for any } L.$$

If $\widehat{\mu}(\xi) = 1$ for some $\xi \in \Xi \setminus \{\mathbf{0}\}$, then $\mu = \delta_{\mathbf{0}}$ by Lemma 4.2. Otherwise $\mu = \mu_{1/p}$. This completes the proof.

5 Limiting measures for rule 90

For the case $p = 2$, the Fourier transform is given by

$$\widehat{\mu}(\xi) = F(\mu)(\xi) = \int_\Omega (-1)^{\langle \xi, \omega \rangle} \mu(d\omega),$$

and the Fourier inversion formula (Theorem 3.1) becomes

$$\mu([\sigma_{x_0} \cdots \sigma_{x_1}]_{x_0}^{x_1}) = \frac{1}{2^{x_1 - x_0 + 1}} \sum_{\xi \in \Xi_{x_0, x_1}} (-1)^{-\sum_{x=x_0}^{x_1} \xi(x)\sigma_x} \widehat{\mu}(\xi). \tag{3}$$

By (2), we have

Lemma 5.1. $\widehat{\beta_\rho}(\xi) = (1 - 2\rho)^{\#\xi}$ for any $\xi \in \Xi$. In particular,

$$\widehat{\beta_0}(\xi) \equiv 1, \quad \widehat{\beta_{1/2}}(\xi) = \begin{cases} 1 & (\xi = \mathbf{0}), \\ 0 & (\xi \neq \mathbf{0}), \end{cases} \quad \widehat{\beta_1}(\xi) = (-1)^{\#\xi}.$$

Lemma 5.2. Suppose that μ is a shift-invariant, strong mixing probability measure on $\Omega = \{0,1\}^{\mathbb{Z}}$. If $|\widehat{\mu}(\xi)| = 1$ for some $\xi \in \Xi \setminus \{\mathbf{0}\}$, then $\mu = \beta_0$ or $\mu = \beta_1$.

Proof. We define a probability measure $\mu * \mu$ on Ω by

$$(\mu * \mu)(\boldsymbol{\sigma}) := \sum_{\boldsymbol{\tau} \in \Omega_L} \mu(\boldsymbol{\tau}) \mu(\boldsymbol{\sigma} - \boldsymbol{\tau}) \quad [\boldsymbol{\sigma} \in \Omega_L].$$

Note that $F(\mu * \mu)(\xi) = \mu(\xi)^2$ for any $\xi \in \Xi$. By (3),

$$(\mu * \mu)(0^L) = \frac{1}{2^L} \sum_{\xi \in \Xi_{1,L}} F(\mu * \mu)(\xi) = \frac{1}{2^L} \sum_{\xi \in \Xi_{1,L}} \widehat{\mu}(\xi)^2 \geqq \frac{1}{2^L} \widehat{\mu}(\mathbf{0})^2 = \frac{1}{2^L} > 0 \quad \text{for any } L.$$

We can see that $\mu * \mu$ is also shift-invariant and strong mixing: The shift-invariance is obvious. For any $\boldsymbol{\sigma} \in \Omega_L$ and $\boldsymbol{\sigma'} \in \Omega_{L'}$, as $n \to \infty$,

$$\begin{aligned}
(\mu * \mu)\left([\boldsymbol{\sigma}]_1^L \cap [\boldsymbol{\sigma'}]_{1+n}^{L'+n}\right) &= \sum_{\boldsymbol{\tau} \in \Omega_L, \boldsymbol{\tau'} \in \Omega_{L'}} \mu\left([\boldsymbol{\tau}]_1^L \cap [\boldsymbol{\tau'}]_{1+n}^{L'+n}\right) \mu\left([\boldsymbol{\sigma} - \boldsymbol{\tau}]_1^L \cap [\boldsymbol{\sigma'} - \boldsymbol{\tau'}]_{1+n}^{L'+n}\right) \\
&\to \sum_{\boldsymbol{\tau} \in \Omega_L, \boldsymbol{\tau'} \in \Omega_{L'}} \mu\left([\boldsymbol{\tau}]_1^L\right) \mu\left([\boldsymbol{\tau'}]_1^{L'}\right) \mu\left([\boldsymbol{\sigma} - \boldsymbol{\tau}]_1^L\right) \mu\left([\boldsymbol{\sigma'} - \boldsymbol{\tau'}]_1^{L'}\right) \\
&= \left\{\sum_{\boldsymbol{\tau} \in \Omega_L} \mu\left([\boldsymbol{\tau}]_1^L\right) \mu\left([\boldsymbol{\sigma} - \boldsymbol{\tau}]_1^L\right)\right\} \left\{\sum_{\boldsymbol{\tau'} \in \Omega_L} \mu\left([\boldsymbol{\tau'}]_1^{L'}\right) \mu\left([\boldsymbol{\sigma'} - \boldsymbol{\tau'}]_1^{L'}\right)\right\} \\
&= (\mu * \mu)\left([\boldsymbol{\sigma}]_1^L\right) \cdot (\mu * \mu)\left([\boldsymbol{\sigma'}]_1^{L'}\right).
\end{aligned}$$

By Lemma 4.2, $\mu * \mu = \beta_0$. Since

$$(\mu * \mu)(\omega(x) = 0) = \mu(\omega(x) = 0)^2 + \mu(\omega(x) = 1)^2 \quad \text{and} \quad \beta_0(\omega(x) = 0) = 1,$$

we can conclude that

$$\mu(\omega(x) = 0) = 1 \text{ or } \mu(\omega(x) = 1) = 1.$$

By the shift-invariance, $\mu = \beta_0$ or $\mu = \beta_1$. □

Now we prove Theorem 2.1. Suppose that

$$P = \int_{\mathcal{M}_1} \mu \, d\pi(\mu) \in \operatorname{Conv}(\mathcal{M}_1),$$

where π is a probability measure on $\mathcal{M}_1$. Since $\Lambda\beta_0 = \beta_0$ and $\Lambda\beta_1 = \beta_0$, we can assume that $\pi(\{\beta_0, \beta_1\}) = 0$. For $\xi \in \Xi$, we have

$$\widehat{P}(\xi) = \int_{\mathcal{M}_1} \widehat{\mu}(\xi) \, d\pi(\mu).$$

Suppose that $P_\infty := \lim_{n\to\infty} \Lambda^n P$ exists. Noting that $P_\infty = \lim_{m\to\infty} \Lambda^{2^m} P$, by Lemmata 3.3 and 3.6, and the bounded convergence theorem, we have

$$\widehat{P_\infty}(\xi) = \lim_{m\to\infty} \widehat{P}(\Lambda^{2^m}\xi) = \lim_{m\to\infty} \int_{\mathcal{M}_1} \widehat{\mu}(\Lambda^{2^m}\xi) \, d\pi(\mu) = \int_{\mathcal{M}_1} \widehat{\mu}(\xi)^2 \, d\pi(\mu).$$

On the other hand, since the limiting probability measure P_∞ is Λ-invariant,

$$\widehat{P_\infty}(\xi) = F(\Lambda^n P_\infty)(\xi) = \widehat{P_\infty}(\Lambda^n \xi) = \int_{\mathcal{M}_1} \widehat{\mu}(\Lambda^n\xi)^2 \, d\pi(\mu) \quad \text{for any } n.$$

Substituting $n = 2^m$ and letting $m \to \infty$, we have

$$\widehat{P_\infty}(\xi) = \int_{\mathcal{M}_1} \{\widehat{\mu}(\xi)^2\}^2 \, d\pi(\mu) = \int_{\mathcal{M}_1} \widehat{\mu}(\xi)^4 \, d\pi(\mu),$$

again by Lemma 3.6 and the bounded convergence theorem. Thus we have

$$\int_{\mathcal{M}_1} \{\widehat{\mu}(\xi)^2 - \widehat{\mu}(\xi)^4\} \, d\pi(\mu) = 0.$$

Since $\pi(\{\beta_0, \beta_1\}) = 0$ and $\widehat{\mu}(\xi)^2 - \widehat{\mu}(\xi)^4 \geqq 0$, Lemma 5.2 shows that

$$\pi(\{\mu \in \mathcal{M}_1 : \widehat{\mu}(\xi) = 0\}) = 1 \quad \text{for } \xi \in \Xi \setminus \{\mathbf{0}\}.$$

By Lemma 5.1,

$$\bigcap_{\xi \in \Xi \setminus \{\mathbf{0}\}} \{\mu \in \mathcal{M}_1 : \widehat{\mu}(\xi) = 0\} = \{\beta_{1/2}\}.$$

Noting that $\Xi \setminus \{\mathbf{0}\}$ is a countable set, we have $\pi(\{\beta_{1/2}\}) = 1$. This completes the proof.

6 Concluding remarks

In this paper, we study limiting measures of iterates of addition modulo p cellular automata, starting from strong mixing measures. Our method can be applied to

$$(\mathcal{L}\omega)(x) = \omega(x) + \omega(x+1) \mod p$$

as well. In this case, Lemma 3.2 should be replaced with $\langle \xi, \mathcal{L}^n \omega \rangle = \langle (\mathcal{L}^*)^n \xi, \omega \rangle$, where $\mathcal{L}^*$ is defined by

$$(\mathcal{L}^* \omega)(x) = \omega(x-1) + \omega(x) \mod p.$$

One of future problems is to extend Theorem 2.1 to addition modulo p CA with p odd prime. For addition modulo 3 CA, by Theorem 2.2, $\mu \in \mathcal{M}_1$ is invariant if and only if $\mu = \delta_{\mathbf{0}}$ or $\mu = \mu_{1/3}$. We remark that

$$\frac{1}{2}\delta_{\mathbf{1}} + \frac{1}{2}\delta_{\mathbf{2}} \in \mathrm{Conv}(\mathcal{M}_1)$$

is another invariant measure.

Our method works well for linear rules with equal coefficients, but we have some troubles to treat other linear rules — the simplest one is $a\omega(x-1) + b\omega(x+1) \mod p$ with nonzero $a \neq b$. On the other hand, it is possible to obtain analogous results for some linear rules depending on more than two coordinates. As an example, we shall prove the following theorem for rule 150, which is an improvement of Theorem 1' in [7]:

Theorem 6.1. We consider the transformation $\widetilde{\Lambda}$ of $\Omega = \{0,1\}^{\mathbb{Z}}$ defined by

$$\big(\widetilde{\Lambda}\omega\big)(x) = \omega(x-1) + \omega(x) + \omega(x+1) \mod 2$$

for $\omega \in \Omega$ and $x \in \mathbb{Z}$. Assume that $P \in \mathrm{Conv}(\mathcal{M}_2)$. Then the following three conditions are equivalent to each other:

(i) $\widetilde{\Lambda}^n P$ converges as $n \to \infty$.

(ii) P is $\widetilde{\Lambda}$-invariant.

(iii) $P = \alpha\beta_0 + \alpha'\beta_{1/2} + \alpha''\beta_1$ for some $\alpha, \alpha', \alpha'' \geqq 0$ with $\alpha + \alpha' + \alpha'' = 1$.

Proof. Since $\beta_0, \beta_{1/2}$ and β_1 are $\widetilde{\Lambda}$-invariant, we can see that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Here we show that (i) implies (iii). Suppose that $P = \displaystyle\int_{\mathcal{M}_2} \mu \, d\pi(\mu) \in \mathrm{Conv}(\mathcal{M}_2)$, where π is a probability measure on $\mathcal{M}_2$. We can assume that $\pi(\{\beta_0, \beta_1\}) = 0$. Rule 150 versions of Lemmata 3.2, 3.5 and 3.6 are:

- Let $\xi \in \Xi$ and $\omega \in \Omega$. For any $n = 1, 2, \cdots$, $\langle \xi, \widetilde{\Lambda}^n \omega \rangle = \langle \widetilde{\Lambda}^n \xi, \omega \rangle$.
- For any $m = 0, 1, 2, \cdots$, $\big(\widetilde{\Lambda}^{2^m}\omega\big)(x) = \omega(x - 2^m) + \omega(x) + \omega(x + 2^m) \mod 2$ for $\omega \in \Omega$ and $x \in \mathbb{Z}$ (Lemma 3' in [7]).
- If $\mu \in \mathcal{M}_2$, then $\lim_{m\to\infty} F\big(\widetilde{\Lambda}^{2^m}\mu\big)(\xi) = \widehat{\mu}(\xi)^3$ for any $\xi \in \Xi$.

Suppose that $P_\infty := \lim_{n\to\infty} \Lambda^n P$ exists. As in the proof of Theorem 2.1, we have

$$\int_{\mathcal{M}_2} \{\widehat{\mu}(\xi)^3 - \widehat{\mu}(\xi)^9\} = 0 \tag{4}$$

for any $\xi \in \Xi$. Now we use a trick in the proof of Theorem 1' in [7]: For any finite sequence $\widetilde{\xi}$ of 0 and 1, let

$$\xi_n = \cdots 000\widetilde{\xi}\underbrace{00\cdots0}_{n}\widetilde{\xi}000\cdots \in \Xi.$$

Since $\mu \in \mathcal{M}_2 \subset \mathcal{M}_1$, we have $\lim_{n\to\infty} \widehat{\mu}(\xi_n) = \widehat{\mu}(\cdots 000\widetilde{\xi}000\cdots)^2$. Substituting ξ_n into (4) and letting $n \to \infty$, we have $\displaystyle\int_{\mathcal{M}_2} \big\{\widehat{\mu}(\xi)^6 - \widehat{\mu}(\xi)^{18}\big\}\, d\pi(\mu) = 0$ for any $\xi \in \Xi$. Now we can conclude that $\pi(\{\beta_{1/2}\}) = 1$. $\square$

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References

[1] H. Cai and X. Luo, “Laws of large numbers for a cellular automaton,” Ann. Probab., vol. 21, 1993, pp. 1413–1426.

[2] B. H. de Menibus, Asymptotic behaviour of cellular automata: Computation and randomness, PhD thesis, Aix-Marseille University, 2014.

[3] D. A. Lind, “Applications of ergodic theory and sofic systems to cellular automata,” Phys. D, vol. 10, 1984, pp. 36–44.

[4] A. Maass, “Rigidity results in cellular automata theory: Probabilistic and ergodic theory approach,” available at https://cantorsalta2015.sciencesconf.org/

[5] I. Marcovici, Automates cellulaires probabilistes et mesures spécifiques sur des espaces symboliques, PhD thesis, Université Paris 7, 2013.

[6] M. Miyamoto, “An equilibrium state for a one-dimensional life game,” J. Math. Kyoto Univ., vol. 19, 1979, pp. 525–540.

[7] M. Miyamoto, “Stationary measures for automaton rules 90 and 150,” J. Math. Kyoto Univ., vol. 34, 1994, pp. 531–538.

[8] M. Pivato, “Module shifts and measure rigidity in linear cellular automata,” Ergodic Theory Dynam. Systems, vol. 28, 2008, pp. 1945–1958.

[9] M. Pivato and R. Yassawi, “Limit measures for affine cellular automata,” Ergodic Theory Dynam. Systems, vol. 22, 2002, pp. 1269–1287.

[10] M. Smorodinsky, Ergodic theory, entropy, Lecture Notes in Mathematics, vol. 214, Springer, 1971.

[11] F. Spitzer, Random fields and interacting particle systems, Math. Assoc. America., 1971.

[12] M. Takei, “On limiting measures for a class of one-dimensional linear cellular automata,” Proceedings of 4th International Workshop on Applications and Fundamentals of Cellular Automata (AFCA’16), held in conjunction with CANDAR’16, 2016, pp. 236–242.

[13] A. L. Toom, N. B. Vasilyev, O. N. Stavskaya, L. G. Mityushin, G. L. Kurdyumov, and S. A. Pirogov, “Discrete local Markov systems,” Stochastic cellular systems: ergodicity, memory, morphogenesis, R. L. Dobrushin, V. I. Kryukov, and A. L. Toom, Eds. Manchester University Press, 1990, pp. 1–182.

[14] S. S. Vallander, “One system of automata with local interactions,” Multicomponent random systems, Adv. Probab. Related Topics, vol. 6, Dekker, 1980, pp. 577–587.

[15] L. N. Vasershtein, “Markov processes over denumerable products of spaces describing large system of automata,” Probl. Inf. Transm., vol. 5, 1969, pp. 47–52.

[16] N. B. Vasil’ev, and I. I. Pyatetskii-Shapiro, “The classification of uniform homogeneous networks,” Probl. Inf. Transm., vol. 7, 1971, pp. 340–346.

[17] S. Wolfram, “Statistical mechanics of cellular automata,” Rev. Modern Phys., vol. 55, 1983, pp. 601–644.