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Two Implementations of Real-Time Sequence Generator for $\{n^3 \mid n = 1, 2, 3, \ldots\}$ and Their Comparison

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Abstract

A cellular automaton (CA) is a well-studied non-linear computational model of complex systems in which an infinite one-dimensional array of finite state machines (cells) updates itself in a synchronous manner according to a uniform local rule. A sequence generation problem on the CA model has been studied for a long time and a lot of generation algorithms has been proposed for a variety of non-regular sequences such as $\{2^n \mid n = 1, 2, 3, \ldots\}$, prime, and Fibonacci sequences etc. In this paper, we study a real-time sequence generator for $\{n^3 \mid n = 1, 2, 3, \ldots\}$. In the previous studies, Kamikawa and Umeo(2018) showed that sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$ can be generated in real-time by an eight-state CA. We present a new six-state implementation of real-time sequence generator for $\{n^3 \mid n = 1, 2, 3, \ldots\}$ rather than reducing the internal state of the Kamikawa and Umeo's sequence generator and give a formal proof of the correctness of the generator. In addition, we show the number of state-changes and number of cells of sequence generators, and compare sequence generators.

Keywords: Cellular automata, Real-time sequence generation problem, Parallel algorithm, Computational complexity

1 Introduction

A model of cellular automaton (CA) was originally devised for studying self-reproduction in biological systems by J. von Neumann [11]. Thereafter, the cellular automaton has been studied in many fields such as complex systems, computability theory, mathematics, and theoretical biology.

A sequence generation problem is one of the major topics in the application of CAs. Arisawa [1], Fischer [2], Korec [10], and Kamikawa and Umeo [3–8] studied the sequence generation problem, where a leftmost cell of the array generates an infinite non-regular sequence indicated by an internal state set. In those studies, much attention has been paid to the developments of real-time generation algorithms and their small-state implementations on CAs for specific non-regular sequences.

Here we study a generation algorithm for sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$. In the previous studies, sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$ generation algorithm on the CA was designed by Kamikawa and Umeo [9].

They showed that sequence $\{n^3 \mid n = 1, 2, 3, ...\}$ can be generated in real-time by an eight-state CA and gave a formal proof of the correctness of the generation algorithm. However, the state lower bound of real-time sequence generator has not been revealed.

In this paper, we present a new six-state implementation of real-time sequence generator for $\{n^3 \mid n = 1, 2, 3, \ldots\}$ rather than reducing the internal state of the Kamikawa and Umeo [9]'s sequence generator and give a formal proof of the correctness of the generator. In addition, we show the number of state-changes and number of cells of sequence generators, and compare six- and eight-state sequence generators. Our motivation to study the sequence generation problem on CAs is to want to show computing power of CAs. Also, it is known that prime, and Fibonacci sequences, and so on appear in various natural phenomena. For example, the Fibonacci sequence appears in biological modelings such as the number of petals in a flower, branching in trees, and the family tree of honeybees. Our sequence generation algorithms would be useful in the simulation and modeling biological pattern formations using CAs.

2 Sequence Generation Problem

A cellular automaton consists of a semi-infinite array of identical finite state automaton, each located at a positive integer point (See Figure 1).

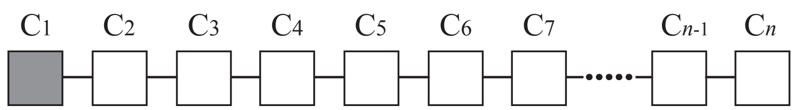

Figure 1: One-dimensional cellular automaton.

Each automaton is referred to as a cell. A cell at point i is denoted by C_i, where $i \geq 1$. Each C_i, except for C_1, is connected to its left- and right-neighbor cells via a communication link. Each cell can know a state of its left- and right-neighbor cells via the communication link. One distinguished leftmost cell C_1, the communication cell, is connected to outside and C_2.

Formally, a cellular automaton (abbreviated by CA) consists of a semi-infinite array of finite state automaton $M = (Q, \delta, \mathtt{b}, \mathtt{a})$, where

1. Q is a finite set of internal states.

2. δ is a transition function defining the next state of a cell such that δ: $Q \times Q \times Q \to Q$, where $\delta(\mathtt{w}, \mathtt{x}, \mathtt{y}) = \mathtt{z}$, $\mathtt{w}$, $\mathtt{x}$, $\mathtt{y}$, $\mathtt{z} \in Q$, has the following meaning: Let t be an integer such that $t \geq 0$. We assume that at step t the cell C_i ($i \geq 2$) is in state $\mathtt{x}$, the left cell C_{i-1} is in state $\mathtt{w}$ and the right cell C_{i+1} is in state $\mathtt{y}$. Then, at the next step $t+1$, C_i takes state $\mathtt{z}$. The leftmost cell C_1 always gets a special state $\$$ from its outside as the state of its left cell. A quiescent state $\mathtt{q} \in Q$ has a property such that $\delta(\mathtt{q}, \mathtt{q}, \mathtt{q}) = \mathtt{q}$ and $\delta(\$, \mathtt{q}, \mathtt{q}) = \mathtt{q}$.

3. A state $\mathtt{b}$ is a special state in Q which C_1 takes at the initial configuration.

4. A state $\mathtt{a}$ is a special state in Q to specify a designated state of C_1 in the definition of sequence generation.

Here we introduce some notations. A transition rule $\delta(w, x, y) = z$ is simply expressed as `w x y` $\to$ `z`. To denote a configuration on a cellular array of length n at time t, we use the following convention: t : $\mathtt{S}_1^t$ ... $\mathtt{S}_n^t$, where $\mathtt{S}_i^t$ denotes the state of the ith cell C_i at time t, $1 \leq i \leq n, t \geq 0$. For convenience, a notation t : $\overbrace{\mathtt{S}...\mathtt{S}}^{[i,j]}$ is also used to denote a partial configuration on neighboring $j - i + 1$ cells, starting from the ith cell C_i to C_j, all in state $\mathtt{S}$ at time t.

We define a new symbol $\Rightarrow$ that shows a synchronous updating of one configuration to the next one with simultaneous applications of the transition rule to each cell. For example, a one-step state transition of M is shown as follows:

$$t : \overbrace{\mathtt{S}_1^t ... \mathtt{S}_n^t}^{[1,n]} \Rightarrow t+1 : \overbrace{\mathtt{S}_1^{t+1} ... \mathtt{S}_n^{t+1}}^{[1,n]}.$$

We now define the sequence generation problem on CA. Let M be a CA, j be a natural number such that $j \geq 1$, and $\{t_n \mid n = 1, 2, 3, \ldots\}$ be an infinite monotonically increasing positive integer sequence defined on natural numbers such that $t_n \geq n$ for any $n \geq 1$. We have a semi-infinite array of cells, shown in Figure 1, and all cells, except for C_1, are in a quiescent state $\mathtt{q}$ at time $t = 0$. The communication cell C_1 takes a special state $\mathtt{b}$ in Q at time $t = 0$ for initiation of the sequence generator. We say that M generates a sequence $\{t_n \mid n = 1, 2, 3, \ldots\}$ in $j\cdot$ *linear-time* if and only if the leftmost end cell of M falls into a special state $\mathtt{a}$ in Q at time $t = j \cdot t_n$. Note that M generates the nth term of t_n at time $t = j \cdot t_n$. In particular, when $j = 1$, we call M a *real-time* sequence generator. In this case, M generates a sequence $\{t_n \mid n = 1, 2, 3, \ldots\}$ without any time delay. Therefore, when $j = 1$, M is optimal in generation steps.

3 Real-Time Sequence Generator for $\{n^3 \mid n = 1, 2, 3, \ldots\}$

In this section, we present two implementations of real-time sequence generator for $\{n^3 \mid n = 1, 2, 3, \ldots\}$. First, we demonstrate a six-state sequence generator. Secondly, we explain an overview of an eight-state sequence generator shown by Kamikawa and Umeo [9]. Lastly, we study the number of state-changes and the number of cells of two sequence generators.

3.1 Six-State Real-Time Sequence $\{n^3 \mid n = 1, 2, 3, ...\}$ Generator

We propose a six-state cellular automaton that can generate sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$ in real-time and give a formal proof for the correctness of the transition rule set proposed.

3.1.1 Space-Time Diagram

We give a sketch of the generation algorithm. The algorithm is described in terms of six signals which propagate at various speeds in the cellular space. We call them waves. They are *a-wave*, *b-wave*, *c-wave*, *d-wave*, *e-wave* and *f-wave*, respectively. See Figure 2 that illustrates a space-time diagram for the real-time generation of the sequence. The propagation speed and direction of each wave in a space-time domain is as follows:

- *a-wave*: 1/1-speed, right,
- *b-wave*: 1/1-speed, left,
- *c-wave*: 0-speed, stationary (marker),
- *d-wave*: 0-speed, stationary (marker),
- *e-wave*: 1/1-speed, left, and
- *f-wave*: 1/1-speed, left.

A rough sketch of the sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$ generation algorithm is as follows:

1. The c-wave, generated by the cell C_2 at time $t = 1$, keeps a marker.
2. The a-wave starts to move in the right direction at time $t = 1$ towards the marker at 1/1 speed, then bounces back at its meeting with the marker to the left end as a b-wave. At the arrival of the b-wave at left end of the array, the a-wave is generated. The a- and b-waves reciprocate between the cell C_1 and the marker. When the a-wave reaches the marker at time $t = 6$, the marker moves to the cell C_3 , and an f-wave is generated on the cell C_2. The f-wave propagates in the left direction, and a d-wave is generated on the cell that the f-wave passed through. The d-wave keeps a marker. The f-wave hits the cell C_1 at time $t = 7$. At the next

step $t = 8$, the cell C_1 falls into the special state a, the a-wave generated, and the cell C_1 generates the next term. Before time $t = 8$, there are not enough cells between the leftmost cell C_1 and the c-wave, so the e-wave is not generated, and the c- and b-waves appear on the cell C_2 simultaneously.

3. When the a-wave conflicts with the d-wave, the a- and d-waves are deleted, and the b-wave is generated. At the arrival of the b-wave at the cell C_1, the a-wave is generated. The a- and b-waves reciprocate until all d-waves are deleted. When the a-wave hits the c-wave, the e-wave is generated. The e-wave propagates in the left direction, and the d-wave is generated on the cell that the e-wave passed through. At the arrival of the e-wave at the cell C_1, the a-wave is generated. The a-, b- and e-waves repeat reciprocating. When a-wave hits the marker for the third time, the marker moves to the right cell, and the d-wave is generated. The f-wave propagates in the left direction, and the cell C_1 falls into the special state a at the next step where the f-wave reaches the cell C_1.

4. Let i be any natural number such that $i \geq 2$. At time $t = i^3$, the c-wave exists on the cell C_{i+1}, d-waves exist on each of cells $C_\ell, 2 \leq \ell \leq i$, the cell C_1 falls into the special state a, and the a-wave generated. The a- and b-waves reciprocate between the cell C_1 and the leftmost d-wave, and the leftmost d-wave is deleted. Then, all d-waves are deleted after the a- and b-waves reciprocate in i times. At the next, the a- and e-waves reciprocate between the cell C_1 and the c-wave existed on the cell C_{i+1}. The e-wave hits the cell C_1 at time $t = i^3 + (2 + 4 + 6 + \ldots + 2i) = i^3 + (i^2 + i)$, therefore the speed of the a- , b- and e-waves are $1/1$. The e-wave then reaches the cell C_1 at time $t = i^3 + 2(i^2 + i)$, after which the f-wave reaches the cell C_1 at time $t = i^3 + 3(i^2 + i)$. At the next step $t = i^3 + 3(i^2 + i) + 1 = (i + 1)^3$, the cell C_1 falls into the special state a.

3.1.2 Implementation

A six-state real-time sequence generator for $\{n^3 \mid n = 1, 2, 3, \ldots\}$ consists of a semi-infinite array of finite state automaton $M = (Q, \delta, \mathtt{b}, \mathtt{a})$, where $Q = \{\mathtt{q}, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$. Table 1 is the transition function $\mathcal{R}_{n^3}$ for the real-time sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$ generator.

Table 1: A six-state transition function $\mathcal{R}_{n^3}$ for real-time generation of sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$.

q	Right State					
Left State	q	a	b	c	d	e
q	q	q	b	c	d	d
a						
b	c					
c	q					
d	q					
e	q					
$	q	q	b	e	e	e

a	Right State					
Left State	q	a	b	c	d	e
q			q	q	q	q
a						
b						
c						
d						
e						
$			q	q		

b	Right State					
Left State	q	a	b	c	d	e
q	b		b			
a			c	c		
b			b	b		
c			b	b		
d						
e						
$	a		a			

c	Right State					
Left State	q	a	b	c	d	e
q			c	c	c	c
a	e		a	a	a	a
b	c					
c	c		c	c	c	c
d						
e			a	a	a	a
$						

d	Right State					
Left State	q	a	b	c	d	e
q	d			d	d	
a	b			c	c	
b						
c	d			d	d	
d	d			d	d	
e	b			c	c	
$						

e	Right State					
Left State	q	a	b	c	d	e
q	e					
a	d					
b						
c	e					
d	e					
e	d					
$				q	q	q

The initial configuration of M at time $t = 0$ is:

$$t = 0 : \overbrace{\mathtt{b}}^{[1]} \; \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[2,\ldots]}.$$

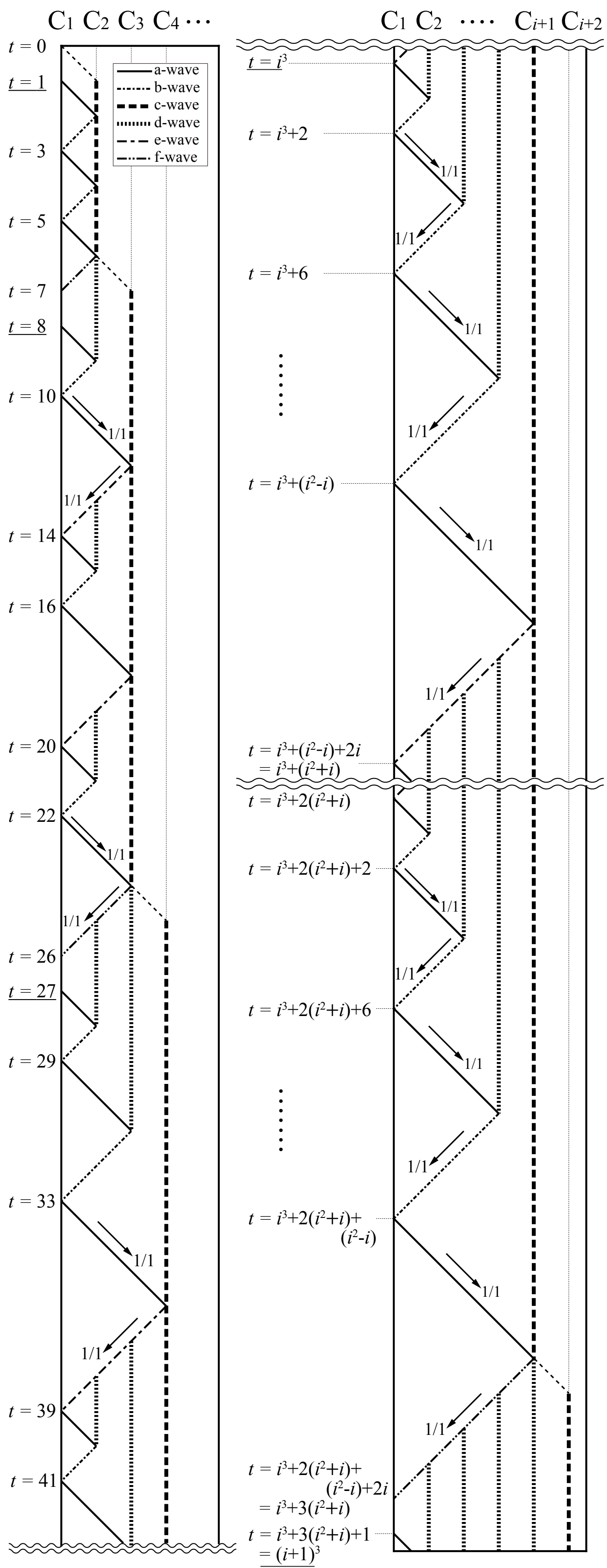

Figure 2: Space-time diagram for real-time generation of sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$.

At time $t = 1$, all cells, except for C_1 and C_2, keep the quiescent state q with the rule q q q $\rightarrow$ q. C_1 and C_2 fall into the state a and c with the rules \$ b q $\rightarrow$ a and b q q $\rightarrow$ c in $\mathcal{R}_{n^3}$, respectively. The configuration at time $t = 1$ is:

$$t = 1 : \overbrace{\texttt{a}}^{[1]}\ \overbrace{\texttt{c}}^{[2]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[3,\ldots]}.$$

At time $t = 2$, all cells, except for C_1, C_2 and C_3, keep the quiescent state q with the rule q q q $\rightarrow$ q. C_1, C_2 and C_3 fall into the state q, e and q with the rules \$ a c $\rightarrow$ q, a c q $\rightarrow$ e and c q q $\rightarrow$ q in $\mathcal{R}_{n^3}$, respectively. The configuration at time $t = 2$ is:

$$t = 2 : \overbrace{\texttt{q}}^{[1]}\ \overbrace{\texttt{e}}^{[2]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[3,\ldots]}.$$

In this way, M takes the following configurations at time $t = 0 \sim 27$.

$$t = 0 : \overbrace{\texttt{b}}^{[1]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[2,\ldots]} \Rightarrow$$

$$t = 1 : \overbrace{\texttt{a}}^{[1]}\ \overbrace{\texttt{c}}^{[2]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[3,\ldots]} \Rightarrow$$

$$t = 2 : \overbrace{\texttt{q}}^{[1]}\ \overbrace{\texttt{e}}^{[2]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[3,\ldots]} \Rightarrow$$

$$t = 3 : \overbrace{\texttt{ee}}^{[1,2]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[3,\ldots]} \Rightarrow$$

$$t = 4 : \overbrace{\texttt{q}}^{[1]}\ \overbrace{\texttt{d}}^{[2]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[3,\ldots]} \Rightarrow$$

$$t = 5 : \overbrace{\texttt{e}}^{[1]}\ \overbrace{\texttt{d}}^{[2]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[3,\ldots]} \Rightarrow$$

$$t = 6 : \overbrace{\texttt{q}}^{[1]}\ \overbrace{\texttt{b}}^{[2]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[3,\ldots]} \Rightarrow$$

$$t = 7 : \overbrace{\texttt{bb}}^{[1,2]}\ \overbrace{\texttt{c}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 8 : \overbrace{\texttt{a}}^{[1]}\ \overbrace{\texttt{b}}^{[2]}\ \overbrace{\texttt{c}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 9 : \overbrace{\texttt{q}}^{[1]}\ \overbrace{\texttt{cc}}^{[2,3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 10 : \overbrace{\texttt{e}}^{[1]}\ \overbrace{\texttt{cc}}^{[2,3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 11 : \overbrace{\texttt{q}}^{[1]}\ \overbrace{\texttt{a}}^{[2]}\ \overbrace{\texttt{c}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 12 : \overbrace{\texttt{qq}}^{[1,2]}\ \overbrace{\texttt{e}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 13 : \overbrace{\texttt{q}}^{[1]}\ \overbrace{\texttt{d}}^{[2]}\ \overbrace{\texttt{e}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 14 : \overbrace{\texttt{e}}^{[1]}\ \overbrace{\texttt{d}}^{[2]}\ \overbrace{\texttt{e}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 15 : \overbrace{\texttt{q}}^{[1]}\ \overbrace{\texttt{c}}^{[2]}\ \overbrace{\texttt{e}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]}$$

$$t = 16 : \overbrace{\texttt{e}}^{[1]}\ \overbrace{\texttt{c}}^{[2]}\ \overbrace{\texttt{e}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 17 : \overbrace{\texttt{q}}^{[1]}\ \overbrace{\texttt{a}}^{[2]}\ \overbrace{\texttt{e}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 18 : \overbrace{\texttt{qq}}^{[1,2]}\ \overbrace{\texttt{d}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 19 : \overbrace{\texttt{q}}^{[1]}\ \overbrace{\texttt{dd}}^{[2,3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 20 : \overbrace{\texttt{e}}^{[1]}\ \overbrace{\texttt{dd}}^{[2,3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 21 : \overbrace{\texttt{q}}^{[1]}\ \overbrace{\texttt{c}}^{[2]}\ \overbrace{\texttt{d}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 22 : \overbrace{\texttt{e}}^{[1]}\ \overbrace{\texttt{c}}^{[2]}\ \overbrace{\texttt{d}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 23 : \overbrace{\texttt{q}}^{[1]}\ \overbrace{\texttt{a}}^{[2]}\ \overbrace{\texttt{d}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 24 : \overbrace{\texttt{qq}}^{[1,2]}\ \overbrace{\texttt{b}}^{[3]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 25 : \overbrace{\texttt{q}}^{[1]}\ \overbrace{\texttt{bb}}^{[2,3]}\ \overbrace{\texttt{c}}^{[4]}\ \overbrace{\texttt{q},\ldots,\texttt{q}}^{[5,\ldots]} \Rightarrow$$

$$t = 26 : \overbrace{\mathtt{b}, \ldots, \mathtt{b}}^{[1,\ldots 3]} \; \overbrace{\;\mathtt{c}\;}^{[4]} \; \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[5,\ldots]} \Rightarrow$$

$$t = 27 : \; \overbrace{\;\mathtt{a}\;}^{[1]} \; \overbrace{\;\mathtt{bb}\;}^{[2,3]} \; \overbrace{\;\mathtt{c}\;}^{[4]} \; \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[5,\ldots]}$$

The overview of the wave generation and its implementation in terms of six states is as follows:

- **a-wave:** The a-wave is depicted by the state `a` or `e`. It is generated on C_1. The state `a` representing the a-wave appears all cells, except for C_1. The state `e` representing the a-wave appears only on C_1. The a-wave propagates in the right direction at 1/1-speed and meets either c- or d-wave which is a stationary state staying on a cell. When the a-wave hits the d-wave, the b-wave which returns to the left direction is generated. On the other hand, when the a-wave hits the c-wave, the e- or f-wave which return to the left direction are generated. See Figure 2.

- **b-wave:** The b-wave is depicted by the state `c`, `d` or `e`. The state `d` and `e` representing the b-wave appear only on C_2 from time $t = 0$ to $t = 7$. At this time, the state `d` and `e` on C_2 represent not only the b-wave but also the c-wave. On the other hand, the b-wave is represented by the propagation of the state `c` after time $t = 7$. The b-wave propagates in the left direction at 1/1-speed and hits the cell C_1. When the b-wave collides with the cell C_1, the a-wave which returns to the right direction is generated.

- **c-wave:** The c-wave is represented by the states `c`, `d` or `e`. The c-wave acts as a marker. When the a-wave hits the c-wave represented by the state `c` or `e`, the e-wave which returns to the left direction is generated, the cell in which the c-wave exists changes from the state `c` to the state `e` or from the state `e` to the state `d`. On the other hand, when the a-wave hits the c-wave represented by the state `d`, the f-wave which returns to the left direction is generated, the c-wave moves to the left cell.

- **d-wave:** The d-wave is represented by the state `b` or `d`. The d-wave acts as a marker. When the a-wave collides with the d-wave, the b-wave which returns to the left direction is generated, the d-wave is deleted.

- **e-wave:** The e-wave is represented by the state `d`. The e-wave propagates in the left direction at 1/1-speed, and the d-wave represented by the state `d` is generated on the cell that the e-wave passed through. The e-wave hits the cell C_1, and the a-wave which returns to the right direction is generated.

- **f-wave:** The f-wave is represented by the state `b`. The f-wave propagates in the left direction at 1/1-speed, and the d-wave represented by the state `b` is generated on the cell that the f-wave passed through. The f-wave hits the cell C_1. At the next step where the f-wave reaches the cell C_1, and the a-wave which returns to the right direction is generated.

We have implemented the rule set in Table 1 on a computer and examined the validity of the table from $t = 0$ to $t = 20000$ steps. In Figure 3, we show a number of configurations in the space-time domain such that $C_i, 1 \le i \le 8, 0 \le t \le 283$.

3.1.3 Correctness

We give a formal proof of the correctness of $\mathcal{R}_{n^3}$. By the definition of the sequence generation problem M takes the following configuration at time $t = 0$:

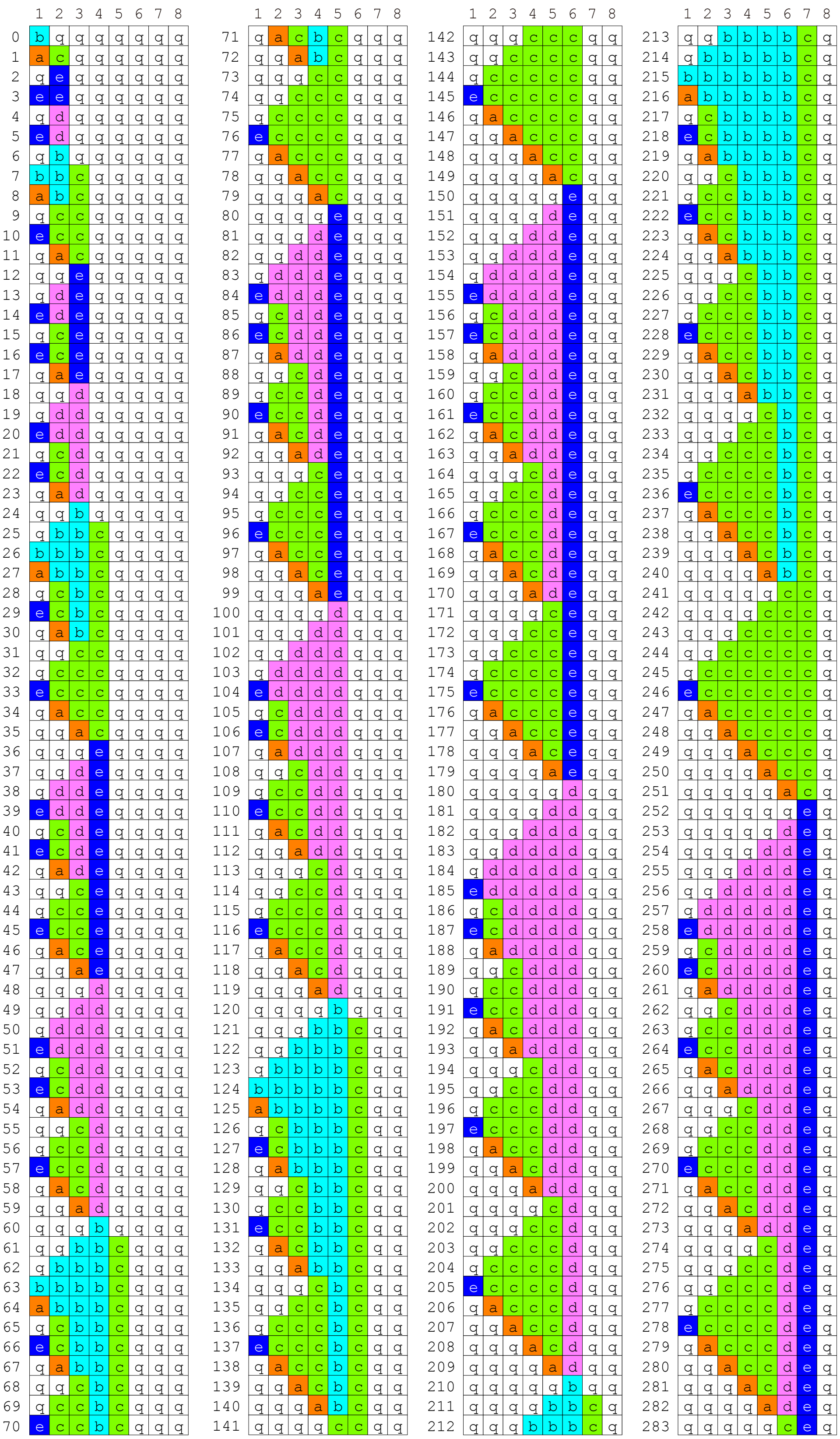

Figure 3: Some configurations of real-time generation of sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$.

$$t = 0 : \overbrace{\mathtt{b}}^{[1]} \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[2,\ldots]}$$

Figure 4 shows snapshots of the generation processes from time $t = 0$ to $t = 6$. These configurations can be obtained by applying the rule set $\mathcal{R}_1$ given in Table 2 to the initial configuration. Note that the rule set $\mathcal{R}_1$ is a subset of $\mathcal{R}_{n^3}$.

	1	2	3	4	5	6	7	8	9	10
0	b	q	q	q	q	q	q	q	q	q
1	a	c	q	q	q	q	q	q	q	q
2	q	e	q	q	q	q	q	q	q	q
3	e	e	q	q	q	q	q	q	q	q
4	q	d	q	q	q	q	q	q	q	q
5	e	d	q	q	q	q	q	q	q	q
6	q	b	q	q	q	q	q	q	q	q

Figure 4: Configurations of sequence generator for $\{n^3 \mid n = 1, 2, 3, \ldots\}$ in the space-time domain such that $\mathrm{C}_i, 1 \le i \le 10, 0 \le t \le 6$.

Table 2: A transition rule set $\mathcal{R}_1$.

\$ a c → q;	\$ q d → e;	\$ q e → e;	d q q → q;	q q q → q;	e e q → d;
e d q → b;	a c q → e;	e q q → q;	b q q → c;	c q q → q;	\$ e d → q;
\$ e e → q;	q e q → e;	\$ b q → a;	q d q → d;		

From Figure 4, we can observe that the cell C_1 falls into the special state a at time $t = 1$ and generates the first terms in real-time. At time $t = 6$, M takes the following configuration:

$$t = 6 : \overbrace{\mathtt{q}}^{[1]} \overbrace{\mathtt{b}}^{[2]} \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[3,\ldots]}$$

Therefore, we obtain the following lemma.

Lemma 1 The first cell C_1 takes the special state a with use of the rule set $\mathcal{R}_1$ at time $t = 1$, that is $\mathtt{S}_1^t = \mathtt{a}$ if and only if $t = 1$ and it can generate the first terms in real-time. The configuration at time $t = 6$ is:

$$t = 6 : \overbrace{\mathtt{q}}^{[1]} \overbrace{\mathtt{b}}^{[2]} \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[3,\ldots]}$$

Next we consider the generation processes at time $t \ge 6$.

Lemma 2 Let i be any natural number such that $i \ge 2$. Let $\mathcal{R}_j, 2 \le j \le 4$ be a rule set given in Tables 3, 4 and 5 , each is a subset of $\mathcal{R}_{n^3}$. M has the following configuration at time $t = i^3 - i$ with use of the rule sets $\mathcal{R}_j, 2 \le j \le 4$:

$$t = i^3 - i : \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[1,\ldots,i-1]} \overbrace{\mathtt{b}}^{[i]} \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[i+1,\ldots]}.$$

Proof
Basis: First, we consider the case $i = 2$. Based on Lemma 1, M takes the following configuration at time $t = 6$ with use of the rule sets $\mathcal{R}_1$ given in Table 2.

$$t = 6 : \overbrace{\mathtt{q}}^{[1]} \overbrace{\mathtt{b}}^{[2]} \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[3,\ldots]}$$

Inductive step: Let k be any natural number such that $k \geq 2$. We assume that M takes the following configuration at time $t = k^3 - k$:

$$t = k^3 - k : \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[1,\ldots,k-1]}\ \overbrace{\mathtt{b}}^{[k]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+1,\ldots]}.$$

M takes the following configurations with use of the rule set $\mathcal{R}_2$ given in Table 3:

$$t = k^3 - k : \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[1,\ldots,k-1]}\ \overbrace{\mathtt{b}}^{[k]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+1,\ldots]} \Rightarrow$$

$$t = k^3 - k + 1 : \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[1,\ldots,k-2]}\ \overbrace{\mathtt{bb}}^{[k-1,k]}\ \overbrace{\mathtt{c}}^{[k+1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+2,\ldots]} \Rightarrow$$

$$t = k^3 - k + 2 : \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[1,\ldots,k-3]}\ \overbrace{\mathtt{b},\ldots,\mathtt{b}}^{[k-2,\ldots,k]}\ \overbrace{\mathtt{c}}^{[k+1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+2,\ldots]} \Rightarrow$$

$$t = k^3 - k + 3 : \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[1,\ldots,k-4]}\ \overbrace{\mathtt{b},\ldots,\mathtt{b}}^{[k-3,\ldots,k]}\ \overbrace{\mathtt{c}}^{[k+1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+2,\ldots]}.$$

Table 3: A transition rule set $\mathcal{R}_2$.

\$ a b → q;	q c b → c;	q c c → c;	\$ q q → q;	\$ q a → q;	\$ q b → b;
q b q → b;	\$ q c → e;	q b b → b;	q a b → q;	q a c → q;	q q q → q;
q q a → q;	q q b → b;	q q c → c;	e c b → a;	e c c → a;	a c q → e;
a c b → a;	a c c → a;	a b b → c;	a b c → c;	b c q → c;	b b b → b;
b b c → b;	b q q → c;	c c q → c;	c c b → c;	c c c → c;	c b b → b;
c b c → b;	c q q → q;	\$ e c → q;	\$ b b → a;		

The state b propagated in the left direction at 1-cell/1-step speed on cell space is called the f-wave, and the state c keeps staying on cell C_{k+1} is called the c-wave. The state b after the f-wave passed through is called the d-wave. The f-wave hits the cell C_1 at time $t = k^3 - k + k - 1 = k^3 - 1$. After time $t = k^3 - 1$, M takes the following configuration:

$$t = k^3 - 1 : \overbrace{\mathtt{b},\ldots,\mathtt{b}}^{[1,\ldots,k]}\ \overbrace{\mathtt{c}}^{[k+1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+2,\ldots]} \Rightarrow$$

$$t = k^3 : \overbrace{\mathtt{a}}^{[1]}\ \overbrace{\mathtt{b},\ldots,\mathtt{b}}^{[2,\ldots,k]}\ \overbrace{\mathtt{c}}^{[k+1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+2,\ldots]} \Rightarrow$$

$$t = k^3 + 1 : \overbrace{\mathtt{q}}^{[1]}\ \overbrace{\mathtt{c}}^{[2]}\ \overbrace{\mathtt{b},\ldots,\mathtt{b}}^{[3,\ldots,k]}\ \overbrace{\mathtt{c}}^{[k+1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+2,\ldots]} \Rightarrow$$

$$t = k^3 + 2 : \overbrace{\mathtt{e}}^{[1]}\ \overbrace{\mathtt{c}}^{[2]}\ \overbrace{\mathtt{b},\ldots,\mathtt{b}}^{[3,\ldots,k]}\ \overbrace{\mathtt{c}}^{[k+1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+2,\ldots]} \Rightarrow$$

$$t = k^3 + 3 : \overbrace{\mathtt{q}}^{[1]}\ \overbrace{\mathtt{a}}^{[2]}\ \overbrace{\mathtt{b},\ldots,\mathtt{b}}^{[3,\ldots,k]}\ \overbrace{\mathtt{c}}^{[k+1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+2,\ldots]} \Rightarrow$$

$$t = k^3 + 4 : \overbrace{\mathtt{qq}}^{[1,2]}\ \overbrace{\mathtt{c}}^{[3]}\ \overbrace{\mathtt{b},\ldots,\mathtt{b}}^{[4,\ldots,k]}\ \overbrace{\mathtt{c}}^{[k+1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+2,\ldots]} \Rightarrow$$

$$t = k^3 + 5 : \overbrace{\mathtt{q}}^{[1]}\ \overbrace{\mathtt{cc}}^{[2,3]}\ \overbrace{\mathtt{b},\ldots,\mathtt{b}}^{[4,\ldots,k]}\ \overbrace{\mathtt{c}}^{[k+1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+2,\ldots]} \Rightarrow$$

$$t = k^3 + 6 : \overbrace{\mathtt{e}}^{[1]}\ \overbrace{\mathtt{cc}}^{[2,3]}\ \overbrace{\mathtt{b},\ldots,\mathtt{b}}^{[4,\ldots,k]}\ \overbrace{\mathtt{c}}^{[k+1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+2,\ldots]} \Rightarrow$$

$$t = k^3 + 7 : \overbrace{\mathtt{q}}^{[1]}\ \overbrace{\mathtt{a}}^{[2]}\ \overbrace{\mathtt{c}}^{[3]}\ \overbrace{\mathtt{b},\ldots,\mathtt{b}}^{[4,\ldots,k]}\ \overbrace{\mathtt{c}}^{[k+1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+2,\ldots]} \Rightarrow$$

$$t = k^3 + 8 : \overbrace{\mathtt{qq}}^{[1,2]}\ \overbrace{\mathtt{a}}^{[3]}\ \overbrace{\mathtt{b},\ldots,\mathtt{b}}^{[4,\ldots,k]}\ \overbrace{\mathtt{c}}^{[k+1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[k+2,\ldots]}.$$

The cell C_1 falls into the special state a at time $t = k^3$. The state a propagated in the right direction at 1/1 speed is called the a-wave, and the state c propagated in the left direction at 1/1 speed is called the b-wave. The a- and b-waves reciprocate until all d-waves are deleted. Then, the a-wave hits the c-wave at time $t = k^3 + k^2$. At this time, M takes the following configuration:

$$t = k^3 + k^2 : \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[1,\dots,k]}\ \overbrace{\mathtt{e}}^{[k+1]}\ \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[k+2,\dots]}.$$

In the next cycle, M transitions with use of the rule sets $\mathcal{R}_3$ given in Table 4. M takes the following configurations:

$$t = k^3 + k^2 : \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[1,\dots,k]}\ \overbrace{\mathtt{e}}^{[k+1]}\ \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[k+2,\dots]} \Rightarrow$$
$$t = k^3 + k^2 + 1 : \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[1,\dots,k-1]}\ \overbrace{\mathtt{d}}^{[k]}\ \overbrace{\mathtt{e}}^{[k+1]}\ \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[k+2,\dots]} \Rightarrow$$
$$t = k^3 + k^2 + 2 : \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[1,\dots,k-2]}\ \overbrace{\mathtt{dd}}^{[k-1,k]}\ \overbrace{\mathtt{e}}^{[k+1]}\ \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[k+2,\dots]} \Rightarrow$$
$$t = k^3 + k^2 + 3 : \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[1,\dots,k-3]}\ \overbrace{\mathtt{d},\dots,\mathtt{d}}^{[k-2,\dots,k]}\ \overbrace{\mathtt{e}}^{[k+1]}\ \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[k+2,\dots]}.$$

Table 4: A transition rule set $\mathcal{R}_3$.

q d d → d;	q d e → d;	q c c → c;	q c d → c;	q c e → c;	$ q q → q;
$ q a → q;	$ q c → e;	$ q d → e;	q a c → q;	q a d → q;	q a e → q;
q q q → q;	q q a → q;	q q c → c;	q q d → d;	q q e → d;	a e q → d;
e d d → c;	e c c → a;	e c d → a;	a d d → c;	a d e → c;	a c c → a;
a c d → a;	a c e → a;	e q q → q;	c e q → e;	c d d → d;	c d e → d;
c c c → c;	c c d → c;	c c e → c;	$ e c → q;	$ e d → q;	d e q → e;
q e q → e;	d d d → d;	d d e → d;			

In this case, the state d propagated in the left direction at $1/1$ is called the e-wave, and the state e keeps staying on cell C_{k+1} is called the c-wave. The state d after the e-wave passed through is called the d-wave. The e-wave hits the cell C_1 at time $t = k^3 + k^2 + k$. After time $t = k^3 + k^2 + k$, the a- and b-waves reciprocate until all d-waves are deleted. Then, the a-wave hits the c-wave at time $t = k^3 + 2k^2 + k$. At this time, M takes the following configuration:

$$t = k^3 + 2k^2 + k : \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[1,\dots,k]}\ \overbrace{\mathtt{d}}^{[k+1]}\ \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[k+2,\dots]}.$$

In the next cycle, M transitions with use of the rule sets $\mathcal{R}_4$ given in Table 5. M takes the following configurations:

$$t = k^3 + 2k^2 + k : \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[1,\dots,k]}\ \overbrace{\mathtt{d}}^{[k+1]}\ \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[k+2,\dots]} \Rightarrow$$
$$t = k^3 + 2k^2 + k + 1 : \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[1,\dots,k-1]}\ \overbrace{\mathtt{dd}}^{[k,k+1]}\ \overbrace{\mathtt{q},\dots,\mathtt{q}}^{[k+2,\dots]} \Rightarrow$$

$$t = k^3 + 2k^2 + k + 2 : \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[1,\ldots,k-2]} \overbrace{\mathtt{d}, \ldots, \mathtt{d}}^{[k-1,\ldots,k+1]} \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[k+2,\ldots]} \Rightarrow$$

$$t = k^3 + 2k^2 + k + 3 : \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[1,\ldots,k-3]} \overbrace{\mathtt{d}, \ldots, \mathtt{d}}^{[k-2,\ldots,k+1]} \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[k+2,\ldots]} .$$

Table 5: A transition rule set $\mathcal{R}_4$.

q d d → d;	q c c → c;	q c d → c;	$ q q → q;	$ q a → q;	$ q c → e;
$ q d → e;	d q q → q;	q a c → q;	q a d → q;	q q q → q;	q q a → q;
q q c → c;	q q d → d;	e d d → c;	a d q → b;	e c c → a;	e c d → a;
a d d → c;	a c c → a;	a c d → a;	c d q → d;	c d d → d;	c c c → c;
c c d → c;	$ e c → q;	$ e d → q;	d d q → d;	d d d → d;	q d q → d;

In this case, the state d propagated in the left direction at 1/1 is called the e-wave, and the state d keeps staying on cell C_{k+1} is called the c-wave. The state d after the e-wave passed through is called the d-wave. The e-wave hits the cell C_1 at time $t = k^3 + 2k^2 + 2k$. After time $t = k^3 + 2k^2 + 2k$, the a- and b-waves reciprocate until all d-waves are deleted. Then, the a-wave hits the c-wave at time $t = k^3 + 3k^2 + 2k = (k + 1)^3 - (k + 1)$. At this time, M takes the following configuration:

$$t = (k + 1)^3 - (k + 1) : \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[1,\ldots,k]} \overbrace{\mathtt{b}}^{[k+1]} \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[k+2,\ldots]} .$$

From the basis and the inductive steps we have Lemma 2. □

Based on Lemmas 1 and 2, the first cell C_1 falls into the special state a by the transition function $\mathcal{R}_{n^3} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4$ at time $t = i^3$ for any $i, i \geq 1$. Thus, we obtain the following theorem.

Theorem 3 Sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$ can be generated in real-time by the six-state cellular automaton with the transition Table 1.

3.2 Eight-State Real-Time Sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$ Generator

We explain an overview of an eight-state sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$ generator shown by Kamikawa and Umeo [9].

3.2.1 Space-Time Diagram

We give a sketch of the generation algorithm. The algorithm is described in terms of six waves. They are *x-wave*, *y-wave*, *p-wave*, *r-wave*, *u-wave* and *v-wave*, respectively. See Figure 5 that illustrates a space-time diagram for the real-time generation of the sequence. The propagation speed and direction of each wave in a space-time domain is as follows:

- *x-wave*: 1/1-speed, right,
- *y-wave*: 1/1-speed, left,
- *p-wave*: 0-speed, stationary (marker),
- *r-wave*: 1/2-speed, right,
- *u-wave*: 1/1-speed, right, and
- *v-wave*: 1/2-speed, right.

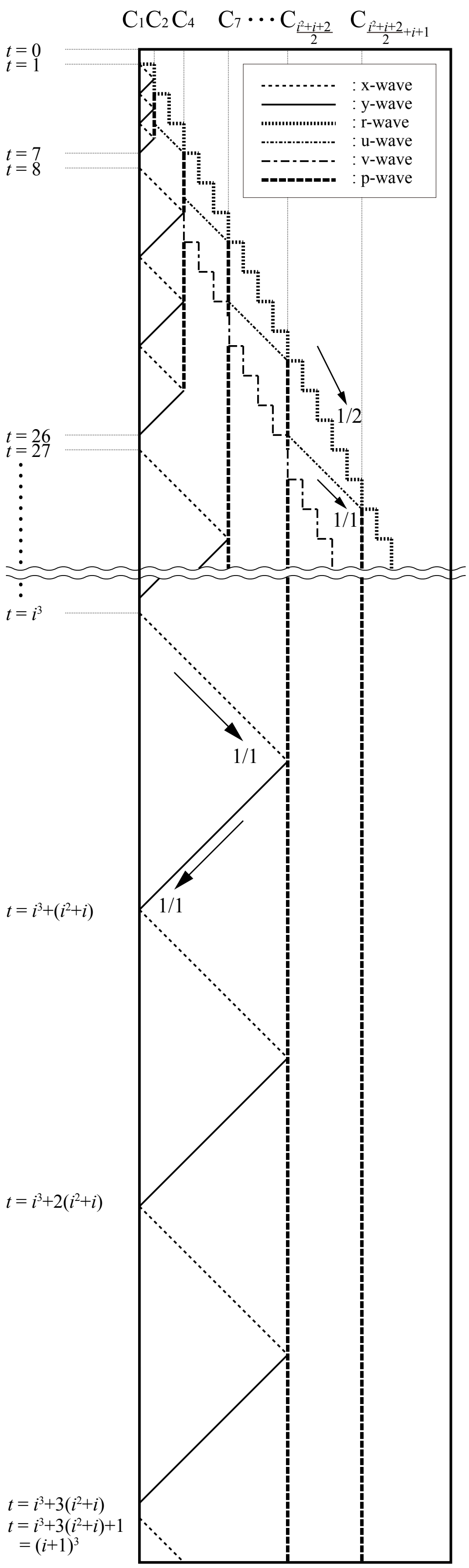

Figure 5: Space-time diagram for real-time generation of sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$.

A rough sketch of the eight-state sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$ generator is as follows:

1. The r-wave, generated by C_2 at time $t = 1$, propagates at 1/2 speed in the right direction.

2. Let i be any natural number such that $i \geq 1$. As the r-, u- and v-waves propagate, p_i-waves are generated on cell C_2, C_4, C_7, C_{11}, ... , $C_{\frac{i^2+i+2}{2}}$, respectively.

3. The x-wave starts to move in the right direction at time $t = 1$ towards the p_1-wave at 1/1 speed, then bounces back at its meeting with the p_1-wave to the left end as a y-wave. At the arrival of the y-wave at left end of the array, the x-wave is generated. The x- and y-waves reciprocate between the cell C_1 and the p_1-wave. When the x-wave reaches the p_1-wave at time $t = 6$, the p_1-wave is disappeared, and the y-wave is generated on the cell C_2. The y-wave propagates in the left direction, and hits the cell C_1 at time $t = 7$. At the next step $t = 8$, the cell C_1 falls into the special state a, the x-wave generated, and the cell C_1 generates the next term. The x- and y-waves reciprocate between the cell C_1 and the p_2-wave.

4. At time $t = i^3$, the p_i-wave exists on the cell $C_{\frac{i^2+i+2}{2}}$, and the x-wave generated. The x- and y-waves make three round trips between the cell C_1 and the p_i-wave. At time $t = i^3 + 3(i^2 + i)$, the y-wave reaches the leftmost cell C_1 for the third time, therefore the speed of the x- and y-waves are 1/1. At the next step $t = i^3 + 3(i^2 + i) + 1 = (i + 1)^3$, the cell C_1 falls into the special state a.

3.2.2 Implementation

An eight-state real-time sequence generator for $\{n^3 \mid n = 1, 2, 3, \ldots\}$ consists of a semi-infinite array of finite state automaton $M = (Q, \delta, \mathtt{c}, \mathtt{a})$, where $Q = \{\mathtt{q}, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}, \mathtt{p}\}$. Table 6 is the transition function $\mathcal{R}_{n^3\_8States}$ for the eight-state real-time sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$ generator.

The initial configuration of M at time $t = 0$ is:

$$t = 0 : \overbrace{\mathtt{c}}^{[1]}\ \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[2,\ldots]}.$$

Table 6: An eight-state transition function $\mathcal{R}_{n^3\_8States}$ for real-time generation of sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$.

q (Left State \ Right State)	q	a	b	c	d	e	f	m
q	q	q	q	q	d	e	f	q
a	a		a	a				
b	q							
c	b							
d	q		q	q				
e	q							
f	f							f
m	q		q	q				
$	q			q	m	m	f	q

a (Left State \ Right State)	q	a	b	c	d	e	f	m
q	m							
a								
b	e		m	e				
c	d							
d	b							
e	e		m	e				
f	f			b				
m	b							
$			b				q	

b (Left State \ Right State)	q	a	b	c	d	e	f	m
q	c					c		c
a	c							
b								
c								
d		b				c		
e								
f		b						c
m	c	b				c		
$				b				c

c (Left State \ Right State)	q	a	b	c	d	e	f	m
q	q	q		q		q		q
a	a							
b	m							
c								c
d						q		
e								
f								
m	q					q		
$	a							f

d (Left State \ Right State)	q	a	b	c	d	e	f	m
q	d	d			d			
a								
b								
c								
d	d		d	d	d			
e								
f								
m	e				m			
$								

e (Left State \ Right State)	q	a	b	c	d	e	f	m
q	e					e		
a								
b		e				e		e
c						b		b
d								
e	e	e				e		e
f								
m	f					m		
$								

f (Left State \ Right State)	q	a	b	c	d	e	f	m
q	f	f					f	
a			c				m	
b								
c								
d								
e								
f	f		f				f	f
m							m	m
$		q					a	

m (Left State \ Right State)	q	a	b	c	d	e	f	m
q	m	m	m	m	q	q	q	q
a								
b	m		m	m				
c	a			a				
d								
e	m			m				
f	m							
m	d							
$					q	q		

At time $t = 1$, all cells, except for C_1 and C_2, keep the quiescent state `q` with the rule `q q q` → `q`. C_1 and C_2 fall into the state `a` and `b` with the rules `$ c q` → `a` and `c q q` → `b` in $\mathcal{R}_{n^3\_8States}$, respectively. The configuration at time $t = 1$ is:

$$t = 1:\ \overbrace{\mathtt{a}}^{[1]}\ \overbrace{\mathtt{b}}^{[2]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[3,\ldots]}.$$

At time $t = 2$, all cells, except for C_1, C_2 and C_3, keep the quiescent state `q` with the rule `q q q` → `q`. C_1, C_2 and C_3 fall into the state `b`, `c` and `q` with the rules `$ a b` → `b`, `a b q` → `c` and `b q q` → `q` in $\mathcal{R}_{n^3\_8States}$, respectively. The configuration at time $t = 2$ is:

$$t = 2:\ \overbrace{\mathtt{b}}^{[1]}\ \overbrace{\mathtt{c}}^{[2]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[3,\ldots]}.$$

In this way, M takes the following configurations at time $t = 0 \sim 27$.

$$t = 0:\ \overbrace{\mathtt{c}}^{[1]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[2,\ldots]} \Rightarrow$$

$$t = 1:\ \overbrace{\mathtt{a}}^{[1]}\ \overbrace{\mathtt{b}}^{[2]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[3,\ldots]} \Rightarrow$$

$$t = 2:\ \overbrace{\mathtt{b}}^{[1]}\ \overbrace{\mathtt{c}}^{[2]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[3,\ldots]} \Rightarrow$$

$$t = 3:\ \overbrace{\mathtt{b}}^{[1]}\ \overbrace{\mathtt{m}}^{[2]}\ \overbrace{\mathtt{b}}^{[3]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 4:\ \overbrace{\mathtt{c}}^{[1]}\ \overbrace{\mathtt{m}}^{[2]}\ \overbrace{\mathtt{c}}^{[3]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[4,\ldots]} \Rightarrow$$

$$t = 5:\ \overbrace{\mathtt{f}}^{[1]}\ \overbrace{\mathtt{a}}^{[2]}\ \overbrace{\mathtt{q}}^{[3]}\ \overbrace{\mathtt{b}}^{[4]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[5,\ldots]} \Rightarrow$$

$$t = 6:\ \overbrace{\mathtt{q}}^{[1]}\ \overbrace{\mathtt{f}}^{[2]}\ \overbrace{\mathtt{a}}^{[3]}\ \overbrace{\mathtt{c}}^{[4]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[5,\ldots]} \Rightarrow$$

$$t = 7:\ \overbrace{\mathtt{ff}}^{[1,2]}\ \overbrace{\mathtt{b}}^{[3]}\ \overbrace{\mathtt{a}}^{[4]}\ \overbrace{\mathtt{b}}^{[5]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[6,\ldots]} \Rightarrow$$

$$t = 8:\ \overbrace{\mathtt{a}}^{[1]}\ \overbrace{\mathtt{f}}^{[2]}\ \overbrace{\mathtt{b}}^{[3]}\ \overbrace{\mathtt{m}}^{[4]}\ \overbrace{\mathtt{c}}^{[5]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[6,\ldots]} \Rightarrow$$

$$t = 9:\ \overbrace{\mathtt{q}}^{[1]}\ \overbrace{\mathtt{cc}}^{[2,3]}\ \overbrace{\mathtt{m}}^{[4]}\ \overbrace{\mathtt{q}}^{[5]}\ \overbrace{\mathtt{b}}^{[6]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[6,\ldots]} \Rightarrow$$

$$t = 10:\ \overbrace{\mathtt{qq}}^{[1,2]}\ \overbrace{\mathtt{c}}^{[3]}\ \overbrace{\mathtt{a}}^{[4]}\ \overbrace{\mathtt{q}}^{[5]}\ \overbrace{\mathtt{c}}^{[6]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[6,\ldots]} \Rightarrow$$

$$t = 11:\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[1,\ldots,3]}\ \overbrace{\mathtt{d}}^{[4]}\ \overbrace{\mathtt{a}}^{[5]}\ \overbrace{\mathtt{q}}^{[6]}\ \overbrace{\mathtt{b}}^{[7]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[8,\ldots]} \Rightarrow$$

$$t = 12:\ \overbrace{\mathtt{qq}}^{[1,2]}\ \overbrace{\mathtt{dd}}^{[3,4]}\ \overbrace{\mathtt{b}}^{[5]}\ \overbrace{\mathtt{a}}^{[6]}\ \overbrace{\mathtt{c}}^{[7]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[8,\ldots]} \Rightarrow$$

$$t = 13:\ \overbrace{\mathtt{q}}^{[1]}\ \overbrace{\mathtt{d},\ldots,\mathtt{d}}^{[2,\ldots,4]}\ \overbrace{\mathtt{b}}^{[5]}\ \overbrace{\mathtt{e}}^{[6]}\ \overbrace{\mathtt{a}}^{[7]}\ \overbrace{\mathtt{b}}^{[8]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[9,\ldots]} \Rightarrow$$

$$t=14: \overbrace{\mathtt{m}}^{[1]}\ \overbrace{\mathtt{d},\ldots,\mathtt{d}}^{[2,\ldots,4]}\ \overbrace{\mathtt{c}}^{[5]}\ \overbrace{\mathtt{e}}^{[6]}\ \overbrace{\mathtt{m}}^{[7]}\ \overbrace{\mathtt{c}}^{[8]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[9,\ldots]} \Rightarrow$$

$$t=15: \overbrace{\mathtt{q}}^{[1]}\ \overbrace{\mathtt{m}}^{[2]}\ \overbrace{\mathtt{dd}}^{[3,4]}\ \overbrace{\mathtt{q}}^{[5]}\ \overbrace{\mathtt{b}}^{[6]}\ \overbrace{\mathtt{m}}^{[7]}\ \overbrace{\mathtt{q}}^{[8]}\ \overbrace{\mathtt{b}}^{[9]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[10,\ldots]} \Rightarrow$$

$$t=16: \overbrace{\mathtt{qq}}^{[1,2]}\ \overbrace{\mathtt{m}}^{[3]}\ \overbrace{\mathtt{d}}^{[4]}\ \overbrace{\mathtt{q}}^{[5]}\ \overbrace{\mathtt{c}}^{[6]}\ \overbrace{\mathtt{m}}^{[7]}\ \overbrace{\mathtt{q}}^{[8]}\ \overbrace{\mathtt{c}}^{[9]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[10,\ldots]} \Rightarrow$$

$$t=17: \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[1,\ldots,3]}\ \overbrace{\mathtt{e}}^{[4]}\ \overbrace{\mathtt{qq}}^{[5,6]}\ \overbrace{\mathtt{a}}^{[7]}\ \overbrace{\mathtt{qq}}^{[8,9]}\ \overbrace{\mathtt{b}}^{[10]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[11,\ldots]} \Rightarrow$$

$$t=18: \overbrace{\mathtt{qq}}^{[1,2]}\ \overbrace{\mathtt{ee}}^{[3,4]}\ \overbrace{\mathtt{qq}}^{[5,6]}\ \overbrace{\mathtt{m}}^{[7]}\ \overbrace{\mathtt{a}}^{[8]}\ \overbrace{\mathtt{q}}^{[9]}\ \overbrace{\mathtt{c}}^{[10]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[11,\ldots]} \Rightarrow$$

$$t=19: \overbrace{\mathtt{q}}^{[1]}\ \overbrace{\mathtt{e},\ldots,\mathtt{e}}^{[2,\ldots,4]}\ \overbrace{\mathtt{qq}}^{[5,6]}\ \overbrace{\mathtt{m}}^{[7]}\ \overbrace{\mathtt{b}}^{[8]}\ \overbrace{\mathtt{a}}^{[9]}\ \overbrace{\mathtt{q}}^{[10]}\ \overbrace{\mathtt{b}}^{[11]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[12,\ldots]} \Rightarrow$$

$$t=20: \overbrace{\mathtt{m}}^{[1]}\ \overbrace{\mathtt{e},\ldots,\mathtt{e}}^{[2,\ldots,4]}\ \overbrace{\mathtt{qq}}^{[5,6]}\ \overbrace{\mathtt{m}}^{[7]}\ \overbrace{\mathtt{b}}^{[8]}\ \overbrace{\mathtt{e}}^{[9]}\ \overbrace{\mathtt{a}}^{[10]}\ \overbrace{\mathtt{c}}^{[11]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[12,\ldots]} \Rightarrow$$

$$t=21: \overbrace{\mathtt{q}}^{[1]}\ \overbrace{\mathtt{m}}^{[2]}\ \overbrace{\mathtt{ee}}^{[3,4]}\ \overbrace{\mathtt{qq}}^{[5,6]}\ \overbrace{\mathtt{m}}^{[7]}\ \overbrace{\mathtt{c}}^{[8]}\ \overbrace{\mathtt{ee}}^{[9,10]}\ \overbrace{\mathtt{a}}^{[11]}\ \overbrace{\mathtt{b}}^{[12]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[13,\ldots]} \Rightarrow$$

$$t=22: \overbrace{\mathtt{qq}}^{[1,2]}\ \overbrace{\mathtt{m}}^{[3]}\ \overbrace{\mathtt{e}}^{[4]}\ \overbrace{\mathtt{qq}}^{[5,6]}\ \overbrace{\mathtt{m}}^{[7]}\ \overbrace{\mathtt{q}}^{[8]}\ \overbrace{\mathtt{b}}^{[9]}\ \overbrace{\mathtt{e}}^{[10]}\ \overbrace{\mathtt{m}}^{[11]}\ \overbrace{\mathtt{c}}^{[12]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[13,\ldots]} \Rightarrow$$

$$t=23: \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[1,\ldots,3]}\ \overbrace{\mathtt{f}}^{[4]}\ \overbrace{\mathtt{qq}}^{[5,6]}\ \overbrace{\mathtt{m}}^{[7]}\ \overbrace{\mathtt{q}}^{[8]}\ \overbrace{\mathtt{c}}^{[9]}\ \overbrace{\mathtt{e}}^{[10]}\ \overbrace{\mathtt{m}}^{[11]}\ \overbrace{\mathtt{q}}^{[12]}\ \overbrace{\mathtt{b}}^{[13]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[14,\ldots]} \Rightarrow$$

$$t=24: \overbrace{\mathtt{qq}}^{[1,2]}\ \overbrace{\mathtt{f},\ldots,\mathtt{f}}^{[3,\ldots,5]}\ \overbrace{\mathtt{q}}^{[6]}\ \overbrace{\mathtt{m}}^{[7]}\ \overbrace{\mathtt{qq}}^{[8,9]}\ \overbrace{\mathtt{b}}^{[10]}\ \overbrace{\mathtt{m}}^{[11]}\ \overbrace{\mathtt{q}}^{[12]}\ \overbrace{\mathtt{c}}^{[13]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[14,\ldots]} \Rightarrow$$

$$t=25: \overbrace{\mathtt{q}}^{[1]}\ \overbrace{\mathtt{f},\ldots,\mathtt{f}}^{[2,\ldots,6]}\ \overbrace{\mathtt{m}}^{[7]}\ \overbrace{\mathtt{qq}}^{[8,9]}\ \overbrace{\mathtt{c}}^{[10]}\ \overbrace{\mathtt{m}}^{[11]}\ \overbrace{\mathtt{qq}}^{[12,13]}\ \overbrace{\mathtt{b}}^{[14]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[15,\ldots]} \Rightarrow$$

$$t=26: \overbrace{\mathtt{f},\ldots,\mathtt{f}}^{[1,\ldots,6]}\ \overbrace{\mathtt{m}}^{[7]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[8,\ldots,10]}\ \overbrace{\mathtt{a}}^{[11]}\ \overbrace{\mathtt{qq}}^{[12,13]}\ \overbrace{\mathtt{c}}^{[14]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[15,\ldots]} \Rightarrow$$

$$t=27: \overbrace{\mathtt{a}}^{[1]}\ \overbrace{\mathtt{f},\ldots,\mathtt{f}}^{[2,\ldots,6]}\ \overbrace{\mathtt{m}}^{[7]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[8,\ldots,10]}\ \overbrace{\mathtt{m}}^{[11]}\ \overbrace{\mathtt{a}}^{[12]}\ \overbrace{\mathtt{qq}}^{[13,14]}\ \overbrace{\mathtt{b}}^{[15]}\ \overbrace{\mathtt{q},\ldots,\mathtt{q}}^{[16,\ldots]}$$

The overview of the wave generation and its implementation in terms of eight states is as follows:

- **x-wave:** The x-wave is depicted by the state `a` or `m`. It is generated on C_1. The state `m` representing the x-wave appears all cells, except for C_1. The state `a` representing the x-wave appears only on C_1. The x-wave propagates in the right direction at 1/1-speed and meets the p_i-wave which is a stationary state staying on the cell $C_{\frac{i^2+i+2}{2}}$. When the x-wave hits the p_i-wave, the y-wave which returns to the left direction is generated.

- **y-wave:** The y-wave is depicted by the state `d`, `e` or `f`. The y-wave propagates in the left direction at 1/1-speed and hits the cell C_1. When the y-wave depicted by the state `d` or `e` collides with the cell C_1, the x-wave which returns to the right direction is generated. On the other hand, the cell C_1 falls into the special state `a` one step after the y-wave depicted by the state `f` arrivals the cell C_1, and the x-wave which returns to the right direction is generated.

- **p_i-wave:** The p_i-wave is represented by the state `m`, `d` or `e`. The p_i-wave acts as a marker. When the x-wave collides with the p_i-wave depicted by the state `m`, the y-wave depicted by the state `d` which returns to the left direction is generated, the p_i-wave changes to the p_i-wave depicted by the state d. Moreover, when the x-wave hits the p_i-wave depicted by the state d, the y-wave depicted by the state e which returns to the left direction is generated, the state constructed the p_i-wave transits the state e. In addition, when the x-wave hits the p_i-wave depicted by the state e, the y-wave depicted by the state f which returns to the left direction is generated, the p_i-wave is deleted.

- **r-wave:** The r-wave is represented by the states b and c. The r-wave propagates in the right direction at 1/2-speed. When the u-wave hits the r-wave, the p_i-wave is generated on the cell $\mathrm{C}_{\frac{i^2+i+2}{2}}$, the r-wave continues to propagate in the right direction.

- **u-wave:** The u-wave is represented by the state a. The u-wave propagates in the right direction at 1/1-speed. When the u-wave collides with the r-wave, the p_i-wave is generated on the cell $\mathrm{C}_{\frac{i^2+i+2}{2}}$, the u-wave is eliminated.

- **v-wave:** The v-wave is represented by the states b and c. The v-wave propagates in the right direction at 1/2-speed. When the v-wave hits the p_i-wave is represented by the state m, the u-wave used to make the p_{i+1}-wave is generated. On the other hand, the v-wave continues to propagate in the right direction after one step delay.

We have implemented the rule set in Table 6 on a computer and examined the validity of the table from $t = 0$ to $t = 20000$ steps. In Figure 6, we show a number of configurations in the space-time domain such that $\mathrm{C}_i, 1 \leq i \leq 35, 0 \leq t \leq 65$.

3.3 Comparison of Sequence Generators

In this section, we compare six- and eight-state sequence generators by the **number of state-changes** and the **number of cells**. Let M be a one-dimensional CA, i be a natural number such that such that $i \geq 1$, a_i be an infinite monotonically increasing positive integer sequence defined on natural numbers such that $a_1 = 1, a_2 = 8, a_3 = 27, \ldots, a_i = i^3$. The **number of state-changes** indicates the total number of changes in the internal state of each cell until time $t = a_i$. In other words, it represents the number of state changes required for sequence generator to generate the i-th term of a_i. The **number of cells** indicates the number of cells necessary for generating the i-th term of a_i. The number of internal states, the number of state-changes, and the number of cells are the evaluation items of sequence generators. Generally, it is considered that sequence generator with less the number of internal states, the number of state-changes and number of cells is better.

3.3.1 Number of State-changes

We revealed number of state-changes of six- and eight-state sequence generators using a computer. In Figure 7, we show number of state-changes of six- and eight-state sequence generators such that a_i, $1 \leq i \leq 12$. Also, the number of state-changes of the six-state sequence generator is $\frac{3i^3+i}{2}$.

3.3.2 Number of cells

So far we have described real-time sequence generation algorithms for $\{n^3 \mid n = 1, 2, 3, \ldots\}$ on CA consisting of the right semi-infinite array of cells. From now on we consider sequence generators consisting of a finite array of cells. Let m be a positive integer such that $m \geq 1$. In this case, CA is constructed of an array of m cells, and the cells are called C_1, C_2, …, C_m from the left edge, respectively. The leftmost cell C_1 always gets a special state \$ from its outside as the state of its left cell, and the rightmost cell C_m always gets a special state \$ from its outside as the state of its right cell. The quiescent state $\mathtt{q} \in Q$ has a property such that $\delta(\mathtt{q}, \mathtt{q}, \mathtt{q}) = \mathtt{q}$, $\delta(\$, \mathtt{q}, \mathtt{q}) = \mathtt{q}$ and $\delta(\mathtt{q}, \mathtt{q}, \$) = \mathtt{q}$. Therefore, a exactly definition of **the number of cells** necessary for time $t \leq a_i$ is the smallest value of m such that the finite array C_1, C_2, …, C_m generates the sequence a_i as desired for time $t \leq a_i$. We show the number of cells of six- and eight-state sequence generators as follows:

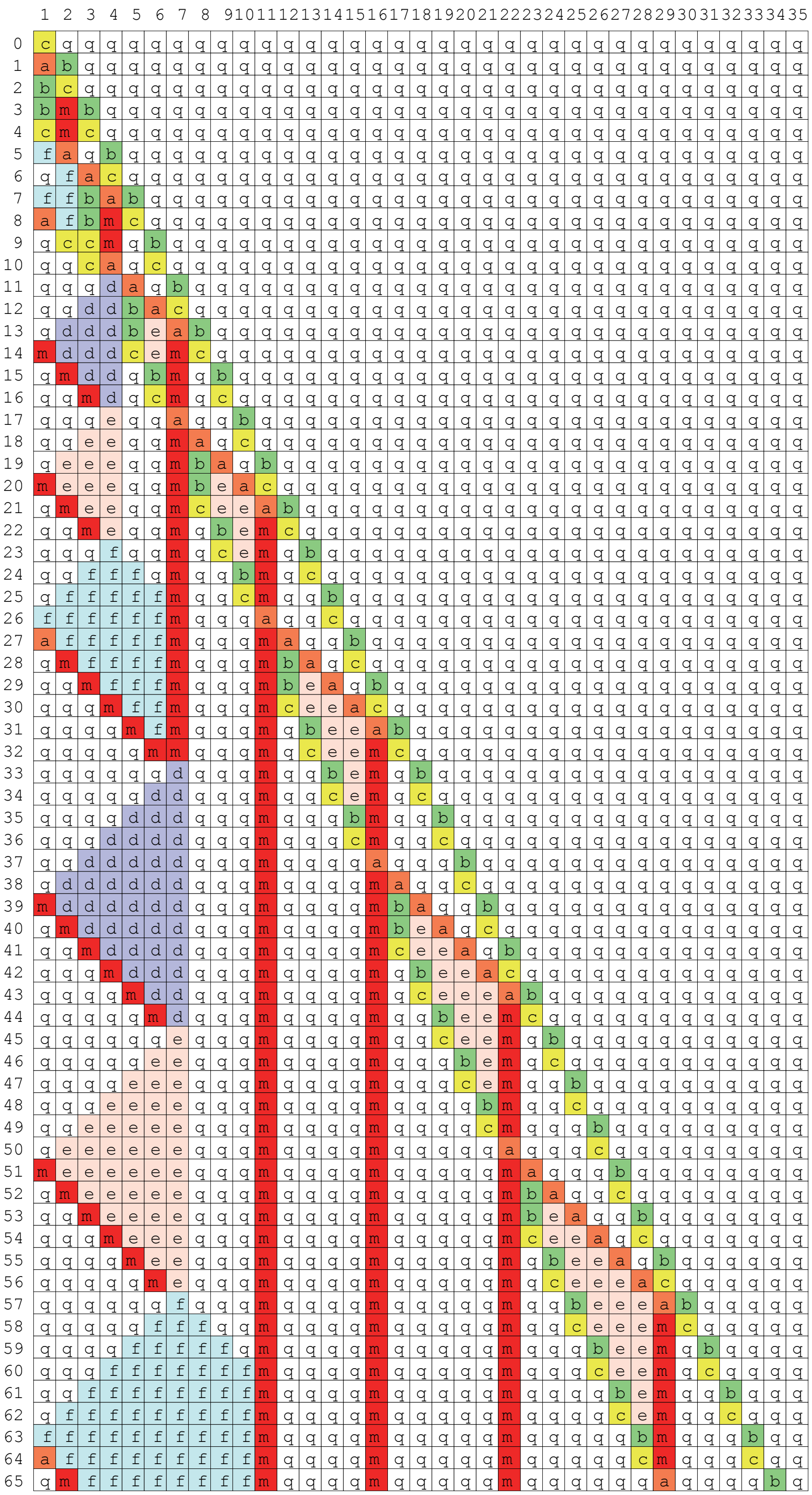

Figure 6: Configurations of real-time generation of sequence $\{n^3 | n = 1, 2, 3, \ldots\}$ in the space-time domain such that C_i, $1 \le i \le 35$, $0 \le t \le 65$.

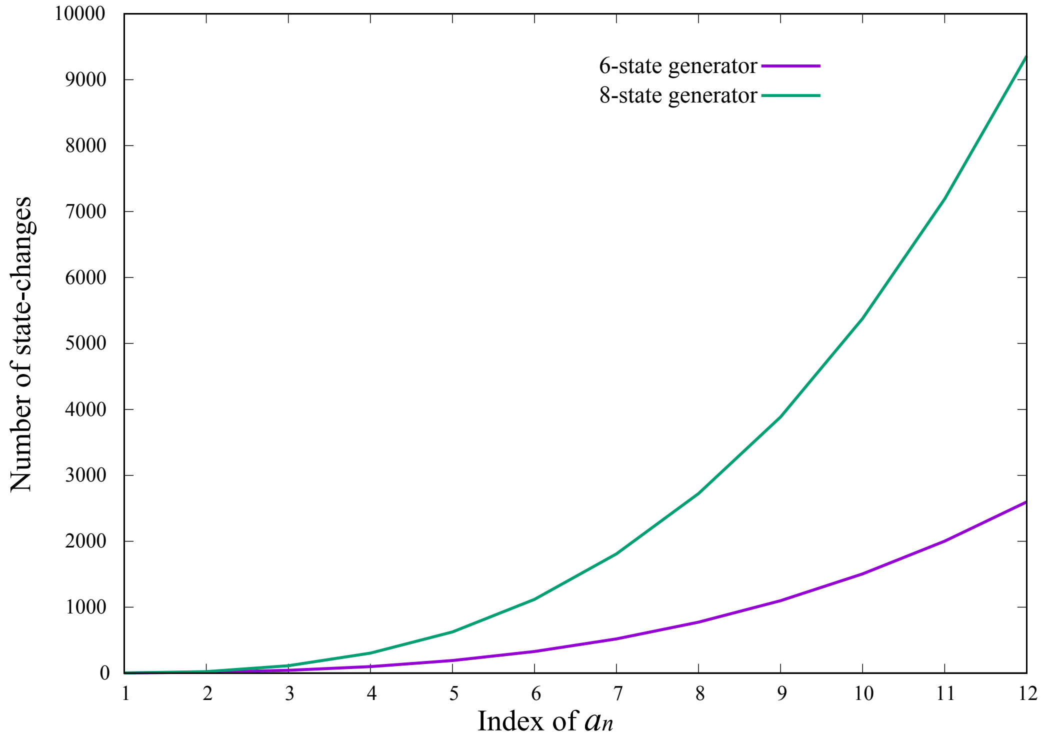

Figure 7: Number of state-changes of six- and eight-state sequence generators.

- **Six-state sequence generator**

  From Lemma 2, M takes the following configurations at time $t = i^3 - i$ and $t = i^3$, respectively.

$$t = i^3 - i : \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[1,\ldots,i-1]} \; \overbrace{\mathtt{b}}^{[i]} \; \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[i+1,\ldots]}$$

$$t = i^3 : \overbrace{\mathtt{a}}^{[1]} \; \overbrace{\mathtt{b}, \ldots, \mathtt{b}}^{[2,\ldots,i]} \; \overbrace{\mathtt{c}}^{[i+1]} \; \overbrace{\mathtt{q}, \ldots, \mathtt{q}}^{[i+2,\ldots]}$$

  When the leftmost cell C_1 falls into a special state $\mathtt{a}$ at time $t = i^3$, the rightmost cell whose state is not a quiescent state $\mathtt{q}$ is the cell C_{i+1}. Therefore, the number of cells of six-state sequence generator is $i + 1$.

- **Eight-state sequence generator**

  The r-wave is generated on the cell C_2 at the time $t = 1$, and proceeds in the right direction at 1/2 speed. See Figure 5. Let N be a set of natural numbers and $P_r(t) : N \cup \{0\} \longrightarrow N$ be a function such that $P_r(t) = \lceil \frac{t}{2} \rceil + 1, t \geq 0$. Then, the r-wave appears on the cell $C_{P_r(t)}$ at time t. Therefore, the number of cells of eight-state sequence generator is $P_r(i^3) = \lceil \frac{i^3}{2} \rceil + 1$.

The number of internal states, the number of state-changes and number of cells of the six-state sequence generator are smaller than the eight-state sequence generator. Thus, we consider that the six-state sequence generator is better than the eight-state sequence generator.

4 Conclusion

A sequence generation problem on CAs has been studied. It has been shown that sequence $\{n^3 \mid n = 1, 2, 3, \ldots\}$ can be generated in real-time by a six-state cellular automaton. We have also given a formal proof of the correctness of the six-state implementation of the algorithm. The number of state-changes and number of cells of six- and eight-state sequence generators have been shown. Our sequence generation algorithms would be useful in the simulation and modeling biological pattern formations using CAs. A further improvement on the number of states and its lower bound would be interesting.

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