

## Algebra of Synchronization with Application to Deadlock and Semaphores

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### Abstract

*Modern multiprocessor architectures have exacerbated problems of coordinating access to shared data, in particular as regards to the possibility of deadlock. For example semaphores, one of the most basic synchronization primitives, present difficulties. Dijkstra defined semaphores to solve the problem of mutual exclusion. Practical implementation of the concept has, however, produced semaphores that are prone to deadlock, even while the original definition is theoretically free of it. This is not simply due to bad programming, but we have lacked a theory that allows us to understand the problem. We introduce a formal definition and new general theory of synchronization. We illustrate its applicability by deriving basic deadlock properties, to show where the problem lies with semaphores and also to guide us in finding some simple modifications to semaphores that greatly ameliorate the problem. We suggest some future directions for deadlock resolution that also avoid resource starvation.*

*Keywords:* synchronization, deadlock, algebra, semaphore

## 1 Introduction

Concurrent programming has become increasingly relevant given the ubiquity of fast networks, multicore processors and highly parallel GPUs (Graphic Processing Units). In all but the most trivial cases, concurrent execution requires coordination between processes - that is, synchronization.

While the application and use of synchronization is ubiquitous, we lack both a comprehensive theory and even a standard definition of synchronization. For example, Kosaraju [15] admits that a precise definition of synchronization is lacking. Things are not much improved since then. Authors on operating systems [24, 23, 20, 19] tend to give examples of synchronization mechanisms, as do

authors on parallel computing [7, 14, 18], but not a formal definition or theory. Apt and Olderog [2] establish a more theoretical approach to synchronization between parallel processes, focusing on the semantics of process waiting for conditions to become true, however they also limit themselves to giving specific examples of what these conditions might be. As a result there is little if any theoretical guidance for understanding synchronization issues. We here introduce a formal definition of synchronization and develop an algebraic theory that allows analysis of synchronization properties. To illustrate the usefulness of our theory, we will apply it in two cases. We will first use it to extend the well-known Coffman deadlock conditions [4]. We will further apply our theory to semaphores, one of the most used synchronization primitives. This paper is an extended version of [9].

## 2 A Mathematical Theory of Synchronization

In order to establish a theory of synchronization, we must first establish the basic ideas of time and synchronization. Note that synchronizations occur in real time; they are therefore defined in terms of an external real clock, independent of times as measured at each process. We take the conventional view of time as measured by clocks which report a single scalar number that is continuously increasing. We assume that time starts sometime before any of the specified events have occurred. We then force processes to wait before executing synchronization code and release the waits in an order satisfying the synchronizations conditions.

We define a simple synchronization as a relation between the times at which two events occur, enforced by code. For example, a barrier is code that forces two events to occur at the same time; a mutual exclusion forces two events to occur at different times; a scheduler enforces an ordering of event times. It is evident from the definition that synchronizations involving multiple events can be built up from simple two event synchronizations.

For example, consider some common synchronizations:

**Barrier** : characterized by events that occur at the same time, trivially all pairs must happen at the same time.

**Mutual Exclusion** : requires that no two events happen at the same time.

**Queues** : an order synchronization in which the events in the set are forced to occur after (or before) the other events in the set. The same applies to baton passing.

This is not meant to be exhaustive, but rather to give a sense for how the pairwise relations are applied in common scenarios.

A synchronization is then characterized by an ordered pair of numbers  $(t_1, t_2)$  and a relation between them, where  $t_1$  corresponds to event 1 and  $t_2$  to event 2. Since  $t_1$  and  $t_2$  are scalars, the set  $\mathbb{R} = \{\geq, \leq, =, \neq, >, <\}$  exhausts all possible relations. We name the synchronization type after the relation in  $\mathbb{R}$ ; so for example we may talk about an “equal” synchronization as the set of pairs  $(t_1, t_2)$  such that  $t_1 = t_2$ ; that is, each synchronization type is given by a set of ordered pairs.

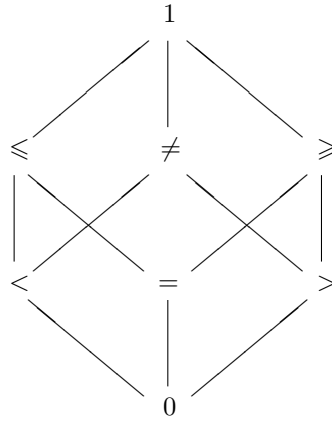
We observe that this definition of synchronization is consistent with practice in that it allows for implementation of any synchronization through forced waits. Assuming a start time before any of the events in a synchronization, we force processes to wait before executing synchronization code and release the waits in an order satisfying the synchronization conditions.

**Theorem 2.1** *If the only restriction on a pair of events is a relation in  $\mathbb{R}$ , and if both events are in the future then it is possible to meet the synchronization condition.*

**Proof:** label the times at which events occur  $t_1$  and  $t_2$ , assume without loss of generality that the conditions for  $t_1$  are met first. Then, if the synchronization is  $t_1 < t_2$  or  $t_1 \leq t_2$  the condition is already met. Otherwise, since time is (by assumption) uniformly increasing, it is sufficient to impose a wait before event 1 as  $t_2$  increases.

**QED**

Figure 1: Partial ordering of  $\mathbb{S}$



The sets that characterize synchronizations are related to each other by various set operations. For example the set of “ $\leq$ ” pairs is the union of the sets “ $<$ ” and “ $=$ ”; the set “ $>$ ” is the intersection of “ $\neq$ ” and “ $\geq$ ”; the sets “ $=$ ” and “ $\neq$ ” are complements of each other. However,  $\mathbb{R}$  is not closed under union and intersection because that the union of all sets in  $\mathbb{R}$  is the set of all pairs, and their intersection is empty (this is also the case for union and intersection of any element and its complement).

If we add the set of all pairs, denoted by “1”, and the empty set denoted by “0” to  $\mathbb{R}$  we obtain the set  $\mathbb{S} = \{1, \geq, \leq, =, \neq, >, <, 0\}$ . We now have that  $\mathbb{S}$  is closed under intersection, union and complement. We further observe that  $\mathbb{S}$  is partially ordered by subset (superset). Under the subset relation, 1 is top, 0 is bottom, and we may graph the partial ordering as a cube. see Figure 1.

In fact,  $\mathbb{S}$  together with union, intersection and complement is a specific instance of a finite algebra of sets, and is therefore a Boolean algebra. (The proof that  $\mathbb{S}$  is closed under union, intersection and complement and is therefore an algebra follows immediately from the fact that the set of relations in  $\mathbb{S}$  is complete for numeric pairs).

A standard definition for a Boolean algebra is a lattice with top and bottom that is distributive and complemented [13]. We will now prove this:

**Theorem 2.2**  $\mathbb{S}$  with union (meet,  $\vee$ ), intersection (join,  $\wedge$ ) and complement ( $\neg$ ) is a Boolean algebra.

**Proof:** Elements of  $\mathbb{S}$  are sets of ordered pairs of scalar numbers. We first note that we have elements 1 which is the set of all such pairs, and 0 which is the empty set. Therefore 1 is a superset of all elements of  $\mathbb{S}$ , and 0 is a subset of all elements.

We identify  $\vee$  with set union  $\cup$  and  $\wedge$  with set intersection  $\cap$ . Therefore  $\vee$  and  $\wedge$  are commutative, associative and idempotent (because the set operations are).

The complement operator  $\neg$  is set complement, it is therefore the case that the union of an element with its complement is 1 (the set of all pairs) and the intersection of an element with its complement is the empty set 0. Further, 1 is an identity for  $\wedge$  and 0 is an identity for  $\vee$ .

By inspection the sets in  $\mathbb{S}$  are closed under  $\vee$ ,  $\wedge$  and  $\neg$ . (Further, since the relations in  $\mathbb{S}$  are complete for pairs of scalars, there is no set of ordered pairs that is not included, so closure is implied).

We note that the subset operator  $\subset$  partially orders elements of  $\mathbb{S}$ , with 1 as top and 0 as bottom. From all the above,  $\mathbb{S}$  with the given operations is a lattice.

$\vee$  and  $\wedge$  are distributive over each other because they are the standard set operators union and intersection, which are distributive.

Therefore  $\mathbb{S}$  with  $\vee$ ,  $\wedge$  and  $\neg$  is a Boolean algebra .

**QED**

Specifically,  $\langle \mathbb{S}, \vee, \wedge, \neg \rangle$  is isomorphic to the Boolean algebra  $\mathbb{B}^3$ , since it is an algebra of subsets with 8 elements [3, 13]. We are not claiming to introduce any new mathematics here; we are simply noting that an old and well developed area of mathematics can be applied to the definition of synchronization.

Where no confusion appears possible we will use the symbol  $\mathbb{S}$  to denote the algebra of synchronization  $\langle \mathbb{S}, \vee, \wedge, \neg \rangle$ .

In general the times that appear in each synchronization pair are selected from an infinite set of numbers. However each synchronization in  $\mathbb{S}$  implies restrictions at least on one of the times in a pair; combinations of these restrictions can make it impossible to satisfy the specific synchronization, leading to deadlock. In algebraic terms, deadlock results when any combination of relations resolves to the relation 0 at any pair of events. It is easy to see how combinations of synchronizations acting on the same event pair can produce deadlock; simply enforce two synchronizations with empty intersection.

The more interesting and practical case results from a set of synchronizations involving multiple events; in this case the restrictions on a particular time  $t_0$  may result from a combination of synchronizations with multiple other events  $t_1, t_2, \dots, t_n$ , each of which may also have synchronizations with each other.

There are a number of ways of getting from the algebra of synchronization to a mathematics of a single event time; we have found it particularly useful to consider that each synchronization in  $\mathbb{S}$  implies bounds on the times at which a specific event may occur.

We find that event times can be unbounded, be bounded from above, from below, or be bounded from above and below. We have therefore four boundedness classes:

Unbounded,  $\bar{B}$ : No limit can be placed on how early or late an event time could occur.

Bounded Above,  $B^+$ : A limit can be placed on how late an event time could occur but not on how early it could occur.

Bounded Below,  $B^-$ : A limit can be placed on how early an event time could occur but not on how late it could occur.

Bounded,  $B$ : Limits can be placed on both how early and how late an event time could occur.

We note that events restricted by  $\neq$  and 1 are unbounded above and unbounded below ( $\bar{B}$ ); relations  $>$ ,  $\geq$  both give bounded below ( $B^-$ ) on the event time on the left, and above ( $B^+$ ) on the event time on the right (the reverse applies to  $<$  and  $\leq$ ).  $=$  and 0 are fully bounded:  $B$  (we consider that upper and lower bounds may coincide in 0, excluding all possible times).

The set  $\mathbb{L} = \{\bar{B}, B^+, B^-, B\}$  is complete, is closed under union, intersection and complement, and is therefore again a Boolean algebra of subsets. In this case  $\langle \mathbb{L}, \wedge, \vee, \neg \rangle$  is isomorphic to  $\mathbb{B}^2$ .

We observe, however, that there are two subalgebras of  $\mathbb{S}$  that induce the algebra  $\mathbb{L}$ , as follows:

1.  $\mathbb{L}_0 = \{\neq, >, <, 0\}$  such that  $\neq$  is top and 0 is bottom.
2.  $\mathbb{L}_1 = \{1, \geq, \leq, =\}$  such that 1 is top and  $=$  is bottom.

(We may think of these as resulting from splitting the cube that represents  $\mathbb{S}$  along the  $=$  axis).

Note that 1 and  $\neq$  do not establish upper or lower bounds,  $>$  and  $\geq$  (likewise  $<$  and  $\leq$ ) establish upper or lower bounds to times on the left or right of the relation,  $=$  and 0 are bounded above and below (in the case of 0, which is the intersection of  $>$  and  $<$ , there is no point that satisfies both boundaries).

An immediate conclusion is that deadlock (0) cannot occur if all relations are in  $\mathbb{L}_1$ , but it can in  $\mathbb{L}_0$ . Unfortunately, many if not most practical synchronization primitives appear to be in  $\mathbb{L}_0$ . Once we include any relation in  $\mathbb{L}_0$  into a set of synchronizations, it becomes possible that an empty interval will result from applying multiple restrictions at a single event. Such an empty interval is deadlock.

### 3 Deadlock

Deadlock is a major problem in parallel programming. Substantial amounts of effort have been dedicated to deadlock detection, which would allow breaking the deadlock so processing can continue. We believe that the wrong problem is being addressed, due to application of deadlock criteria that depend on assumptions not appropriate to parallel execution.

Part of the problem is that there is no agreement on what is meant by deadlock. Tanenbaum [22] characterizes deadlock as a subset  $\Sigma$  of states such that there is no transition out of  $\Sigma$ , and there are no transitions in  $\Sigma$  which cause forward progress. Apt and Olderog, [2] similarly characterize deadlock, as a configuration that has no successor in the state transition relation. A more common characterization is found in Lynch [16], who defines deadlock in terms of a circular dependence between processes. Elmasri and Navathe [6] give a similar definition, as do most database references of which we are aware.

We define deadlock as follows:

**Definition 3.1 (Deadlock)** *when any relation between events in an execution cannot be resolved and thus is in  $\{0\}$ .*

We claim that our definition includes all definitions of deadlock We have:

**Theorem 3.2** *All definitions of deadlock are included within  $\{0\}$*

**Proof:** Suppose false. Then, for some definition of deadlock there is no event such that its conditions can not be met. Therefore execution continues and we have absence of deadlock.

**QED**

We consider now some consequences of our definition. Most authors, for example Silberschatz [19] and Tanenbaum [23] quote the results derived by E. G. Coffman, M. J. Elphick, and A. Shoshani's 1971 paper [4] which are:

1. Mutex
2. Hold and wait
3. No Preemption
4. Circular wait

We may now show how our definition relates to the Coffman conditions.

1. Mutex requires that equality cannot exist between events, and so a group of events are in  $L_0 = \{\neq, <, >, 0\}$ .
2. Hold and wait requires that the process itself will not change the timing constraint.
3. No Preemption requires that no other process can change the timing constraint. Essentially, 2 and 3 require a static timing relation.
4. Circular wait requires a chain of  $<$  or  $>$  which begins and ends with the same event, thus causing that event to be in  $\{0\}$ .

The Coffman conditions are thus one way (even the most common way) of achieving a deadlock, but they are not the only way. For instance, if a process becomes a zombie, or is waiting on a zombie process, it will deadlock because a zombie process is in  $\{0\}$  by definition. There is no need for a circular wait or even mutex in this case. Another way to get deadlock is to write a program which will only run if it is sleeping, thus enforcing  $A \neq A$  and thus  $A \in \{0\}$ .

Our definition also subsumes Tanenbaum's definition, as the requirement for no transitions out implies the event of being in  $\Sigma$  is in either  $\{1\}$  or  $\{0\}$ , and the requirement of no progress implies

it is in  $\{0\}$ . Apt and Oderog's definition only implies that some event is in  $\{1\}$  or  $\{0\}$ , and thus includes things which are not deadlock, such as a program which asserts a power on signal. Such a program would not have a successor state but is still doing a useful action thus progressing, and so would not be considered as deadlocked, being in  $\{1\}$ . The remaining definitions essentially define deadlock as something that meets the Coffman conditions, and so also meets our definition.

This definition allows a clearer understanding of the problem with deadlock, even showing how adding semaphores to a program is dangerous by forcing the program into  $L_0$ , where we are a bad schedule away from deadlocking.

## 4 Semaphores

Semaphores are an extremely useful construct in concurrent programming, used to implement critical sections and in general to keep processes from interfering with each other when accessing shared resources or data. Experience shows, however, that once we get past the simplest uses, in particular when multiple processes must acquire multiple semaphores, semaphores become hard to use. It is difficult in practice to understand the effect of all possible combinations of semaphore access by multiple processes, and deadlock becomes a possibility.

Dijkstra's paper introducing the concept of semaphores [5] proved they were free from deadlock, but practical implementations have added complexity in the form of queues [20, 23]. This adds fairness but introduces the possibility of deadlock. Various other schemes have been proposed - for example "weak" queues and blocked sets [20, 21], multiple semaphore instructions [15, 17, 10], and other more complicated mutual exclusion primitives such as monitors [11, 12]. The most used mutual exclusion semaphore remains the queued semaphore, we suspect this is because of its simplicity and ease of hardware support. Even so, Tanenbaum remarks [24] that even a quarter century after their introduction, research into semaphores continues. This is still true, for example, Agarwal and Stoller [1] examines potential deadlock detection in programs using semaphores.

We will restrict our attention to semaphores as commonly implemented; that is the basic  $P$  and  $V$  primitives used for mutual exclusion, with a FIFO queue holding blocked processes to guarantee fairness [20].

An immediate challenge to our algebraic theory is the syntax of semaphores:

```
P(s) // acquire semaphore s
      // or wait in queue
{
  // critical section code block
}
V(s) // release semaphore s
```

A semaphore establishes an interval during which the  $\neq$  relation holds (mutual exclusion), but we would like to identify a single event that we associate with a semaphore and here we apparently have two events - semaphore acquisition  $P$  and release  $V$ . Even so, mutual exclusions should not be dangerous. Even though they are in the subalgebra  $\mathbb{L}_0$  and therefore admit the possibility of deadlock, any combination involving only union or intersection of  $\neq$  still yields  $\neq$ , which is unbounded.

Problems are introduced by the queues we attach to semaphores to ensure fairness. Queues are an ordering, and thus a  $>$  synchronization. Note that the intersection  $\{\neq\} \wedge \{>\} = \{>\}$ ; therefore a semaphore and queue is an ordering synchronization  $>$ .

Semaphores are implemented as two instructions defining an interval. If a particular process acquires more than one semaphore, it is possible for the corresponding intervals to be nested or overlapped. What does  $>$  mean in this situation? If we need to consider both  $P$  and  $V$  in analyzing the semaphore, then a semaphore  $P_i, V_i$  is a two point synchronization and does not fit in the algebra.

Fortunately, this does not happen, although the point is not clear in the literature. For example, recent editions of major Operating Systems texts such as Stallings [20] pg.227, and Silberschatz and Galvin [19] pg.204 point out a need to carefully analyze or nest the location of semaphore release

instructions. It is possible to prove, by considering multiple cases of different combinations of process acquisition and release of semaphores, that in fact only the semaphore acquisition affects the order in which processes are blocked and released (as long as each semaphore acquisition is matched by a subsequent release in any order).

This fact is easily and elegantly proved in our theory, as follows. An alternate proof that does not rely on our theory is included in Appendix A for comparison.

**Theorem 4.1** *The boundedness class (in  $\mathbb{L}$ ) of event sequences for processes synchronized by a set of semaphores depends only on the  $P$  instructions, not on the  $V$  instructions.*

(i.e.:  $\{P_1, P_2\} \times \{V_2, V_1\} := \{P_1, P_2\} \times \{V_1, V_2\}$ )

**Proof:** Let  $s_i = \{P_i\} \times \{V_i\}$  be a semaphore with  $P_i$  and  $V_i$  the usual acquire and release events. Now when a process tries to acquire the semaphore ( $P_i$ ), it could be blocked or allowed to acquire. If it acquires the semaphore its boundedness class does not change as no new information on the boundedness class is learned, however if the process is blocked then the boundedness class has been intersected with  $\neq$ , if it is also queued then it is intersected with  $<$ . For any process that has the semaphore when another is blocked this causes its boundedness class with respect to the blocked process to be intersected with  $\neq$ . For a process that is queued before the new process becomes queued then based on the priority scheme the boundedness class will be changed for both. It does not matter how it changes (either intersected with  $<$  or  $>$ ), the key point is the  $P_i$  can and often will change the boundedness class of processes synchronized with a semaphore.

When a process has a semaphore, that semaphore does not provide any blocking to it. It can easily release the semaphore ( $V_i$ ) and no process can block this. Some might protest that it might have to acquire another semaphore before it can release the current one, but that is a block on some  $P_j$ , not on  $V_i$ . It is thus apparent that  $V_i$  does not impose any restrictions on the boundedness class.

**QED**

Following the theorem, we can indeed analyze semaphores with queues as “ $>$ ” synchronizations relating the  $P$  semaphore acquisition events, which determine process position in the semaphore queues. In the following theorem, we have generalized from semaphores to synchronizations which impose orderings  $<$  or  $>$  if in reverse order, which includes any semaphore with fairness conditions.

**Theorem 4.2** *In situations where only a common set of synchronizations,  $S$  with cardinality of 2 or greater, imposes orderings  $<$  (or  $>$  if in reverse order), the following holds: There is a schedule that results in deadlock if and only if it is possible for processes to access elements of  $S$  in different orders.*

**Proof:** (If) Assume there is a schedule which results in deadlock, thus we are in 0. The synchronizations can only induce  $<$  (or  $>$  if they were in inverted order). It follows that the only way to obtain deadlock is  $\{<\} \wedge \{>\} = 0$ . Thus some process must acquire at least two synchronizations in the opposite order of some other process.

(Only If) Let  $q_1$  and  $q_2$  be two processes which share the set of synchronizations  $S$ . Since the cardinality of  $S$  is at least two, then there are at least two synchronizations  $s_1 = P_1 \times V_1$  and  $s_2 = P_2 \times V_2$  in  $S$ . Without loss of generality, consider the schedule, where  $q_1$  seeks to acquire  $s_1$  then  $s_2$  and  $q_2$  seeks to acquire  $s_2$  then  $s_1$ . Thus due to  $q_1$  we have  $s_1 < s_2$ , but from  $q_2$  we have  $s_1 > s_2$ . We thus have  $\{<\} \wedge \{>\} = 0$ , and deadlock is possible.

Note that boundedness classes create necessary conditions on synchronizations. Sufficiency requires that  $q_1$  acquire  $s_1$  before  $q_2$  and  $q_2$  acquire  $s_2$  before  $q_1$ . We only need to show existence, and since there is no atomicity between  $P_1$  and  $P_2$ , consider the schedule where  $q_1$  acquires  $s_1$ , then is interrupted, and  $q_2$  woken, then it acquires  $s_2$ . Thus there exists at least one schedule, which results in deadlock if permutations of synchronization acquisition is allowed.

**QED**

That is, if any process acquires more than one semaphore, it is asserting  $=$  (for the mutexes protected by each semaphore) and  $\neq$  (the semaphores themselves) at the same time - this is the worst possible case, since the intersection  $\{=\} \wedge \{\neq\}$  is empty! If another process also asserts both semaphores, binds them to  $=$  at a different time, we can get deadlock. This is immediately evident from our algebra: Since mutual exclusion is in  $\mathbb{L}_0$ , anything that enforces it implies the possibility of deadlock.

The analysis of the problem with semaphores we have described is not dependent on the algebra of synchronization, in that it can be shown by other means, although the algebra has guided us to find and prove it more easily. The algebra, however, allows us to prove that the problem is inescapable - anything we do that involves mutual exclusion and multiple queues will introduce at least the possibility of deadlock. We may here object that Dijkstra proved both absence of deadlock and fairness properties for his semaphores in [5]. However in Dijkstra's scheme it is not possible for two different processes to assert  $P$  on the same semaphore, therefore Dijkstra semaphores do not meet the criteria our deadlock proofs, but also are too restrictive for practical implementation.

#### 4.1 How To Fix Semaphores

Dijkstra's original concept of semaphores, in which semaphore are owned by processes and could only be set by owner, coupled with no queue or other structure, puts semaphores in  $\neq$  instead of  $<$  or  $>$ . Such a semaphore would not deadlock on its own, provided the acquisition of the semaphore was not protected by the same semaphore.<sup>1</sup> Multiple Dijkstra semaphores would just be  $\neq$ , since  $\{\neq\} \wedge \{\neq\} = \{\neq\}$ . Fairness considerations and other desirable system goals can impose conditions on the boundedness classes of the semaphores and result in forcing the system to null if the order of one semaphore acquisition is ever inverted (cycle). An immediate fix is to force acquisition of semaphores in a global order by:

- runtime system like SOS [8].
- DLL linked to OS and compiler.

This fix would work on current code without recompilation. Slightly more involved but superior fixes include:

- force process to release semaphores it holds if made to wait. This is unwieldy at the OS level, because if queues are length  $M$  and there are  $N$  semaphores, it is  $O(N * M * M)$  (check all queues and compress as needed)
- bounce processes into random wait (like ethernet) and force semaphore release - better, since if there are no queues, work by OS is  $O(N)$ . However this remedy introduces the possibility of starvation.

None of these are perfect, which underscores why there continues to be interest in semaphore research.

In fact, we are now in a position to state that, without additional restrictions, the goals of mutual exclusion, fairness and deadlock freedom can not be achieved at the same time. First, we need to define fairness.

**Definition 4.3 *Fairness.***

*Fairness requires that a process requesting a resource:*

1. *Eventually acquire the resource*
2. *Acquire the resource no later than other processes that made later requests for the same resource.*

---

<sup>1</sup>this would assert  $p \neq p$  which is trivially 0.



(1) is a no-starvation condition; (2) enforces a ‘no cutting ahead in the line’ principle. Even though condition (2) appears to imply (1), we could have a limiting case in which nobody gets the resource.

Putting this in the context of our theory, we can read ‘request a resource’ as ‘requests a future occurrence of an event in which the resource is acquired’. We also interpret resource to include any external thing a process might need to continue execution, including things like receipt of a message or the availability of an external process to receive a message. Fairness thus imposes  $\leq$  or  $\geq$ .

We can now state the theorem.

**Theorem 4.4** *Mutual exclusion, freedom from deadlock and fairness can not be achieved in a system that has more than a single resource that must be accessed, and in which it is possible that processes will request more than one resource at the same time, without imposing additional restrictions on synchronizations.*

**Proof:**

Assume there are two resources  $r_1$  and  $r_2$  that require mutual exclusion ( $\neq$ ).

Now let  $p$  and  $q$  be processes, each of which requires access to  $r_1$  and  $r_2$  simultaneously. Then we have 4 events:  $p_1, p_2, q_1$  and  $q_2$ ; given simultaneous access we also have  $p_1 = p_2$  and  $q_1 = q_2$ .

Since resources may be requested without restriction by each process, then there is a schedule in which conditions

1.  $p$  requests  $r_1$  first,  $(p_1 \leq q_1)$ , thus  $\{\leq\}$ ,
2.  $q$  requests  $r_2$  first,  $(q_2 \leq p_2) = (q_1 \leq p_1) = (p_1 \geq q_1)$ , thus  $\{\geq\}$ ,
3.  $r_1$  and  $r_2$  has mutual exclusion, thus  $(p_1 \neq q_1), (p_2 \neq q_2)$ , thus  $\{\neq\}$ ,

are asserted at the same time, thus according to the algebra of synchronization,

$$\{\leq\} \wedge \{\geq\} \wedge \{\neq\} = \{0\}. \tag{1}$$

Thus, there exists a program, which deadlocks. Therefore deadlock is possible if we have fairness and mutual exclusion, absent additional restrictions.

**QED**

Note that this does not mean every system will deadlock with mutual exclusion and fairness, but rather that deadlock is possible and thus cannot be ignored.

We now consider ‘interesting’ systems of processes which assert synchronizations; by which we mean realistic systems of processes which are subject to both finite resources and which include multiple events.

We have from theorem 4.4 that if we impose fairness and mutual exclusion without further restrictions, our system is not free from deadlock.

## 4.2 Alternatives to semaphores

A question we need to ask is should we fix semaphores. Semaphores are not just a construct that requires careful programming - once we are out of Dijkstra’s original concept, semaphores are in theory an inherently dangerous construct - and not even fully protective, since resources may be grabbed without checking semaphore. We need to consider if additional restrictions or limitations we impose on semaphores are in fact worthwhile.

A natural objection arises to not throw out a common tool that has been used so long due to flaws. In response we certainly acknowledge the great history of the semaphore, but note the flaws are fundamental to the construct. A semaphore either lacks fairness, can deadlock, or requires some global mitigation scheme (ordering, forced release, etc.) which violates the distributed nature of semaphores. There is no avoiding the possibility. Do we wish to continue to use a flawed concept due to history?

A second objection arises as to what alternatives exist. We note that with regard to resource access - processes don't actually need to grab resource, they just need access to a resource queue. Each process can have a locally owned queue that signals the OS when it has items in it, then OS can check the process queues, order the resource requests, and pass requests to global queue - this is like a queue of queues, or a multi-tailed queue. For normal resource access, reads are not an issue so you only need this for writes, though a combined system could be done if needed. This scheme has a number of implications:

- **ALGEBRAIC IMPLICATIONS** - since the multi-tailed queue is global and both ordered and maintained by the OS, it imposes a consistent ordering synchronization. Theorem 4.2 thus shows that such a multi-tailed queue cannot deadlock, since queues impose orderings  $<$  or  $>$  if in reverse order, but the globally ordered queue does not permit reverse order, so no schedule can deadlock. This corresponds to uniqueness of resource - semaphores are a distributed construct, no particular reason they should lead to any consistent ordering on a single shared resource.
- **DESIGN PRINCIPLE** - cardinality of control should match cardinality of resource - if single resource, access control should be single as well.
- **DB** - database needs to sequence reads, as well as writes - multi-tailed queues still work, but now they include writes. Note certain circumstances might require priority queues.
- **REALTIME** - requires hard allocation to meet time constraints, so the multi-tailed queue will not work, but this does not work under semaphores either.
- **SCALABILITY** - in a distributed system, each node handles its own queues which are global to the node. Only problem is inter-node requests.
- **INTER-NODE** - if all inter-node requests are handled as messages, then an SOS system message queue resolves all scheduling conflicts, see [8]. The only problem not handled by this is a data cycle.
- **DATA CYCLES** - can be broken by assuming values, converge over multiple iterations to consistent values (standard numeric technique). Problem only arises in inter-node message cycles, which could be detected by SOS-like system (since SOS knows the variables it is communicating - high level semantics interaction to runtime system). If OS can detect cycle, OS can schedule a thread for iterative solution.

A multi-tailed queue is thus a feasible solution to synchronizations in most areas without requiring semaphores.

However a multi-tailed queue is not fair, since the OS is imposing a global ordering and does not permit the resources to be requested out of this order. This forces a reasonably close approximation to fairness, but not true fairness, as the one requesting something out of the system's prescribed order would be forced to wait for the one that requested in order. Also, the imposition of a centralized global queue is a major additional restriction, with the attendant challenges. It does however have the virtue of avoiding the problems of an unrestricted case as was shown in for example theorem 4.2. Theorem 4.2 applies in the unrestricted case, but leaves open the possibility that additional restrictions on execution may resolve the problem, which is what is being exploited in the multi-tailed queue, and serves as an example of what can be done to get around the problem.

## 5 Conclusions and Continuing Work

We have introduced a formal definition and algebra of synchronization, which permits the analysis of when deadlock can occur. We followed with an analysis showing that the four traditional deadlock conditions follow from our algebra, but in fact describe a subset of the problem.

We then analyzed semaphores and showed why current implementations fail, while Dijkstra's version does not. We then presented fixes and an alternative that will not deadlock.

We are expanding the concept to include data cycles and solutions to this problem, which were introduced briefly here and in [8].

The algebra of synchronization has not yet been applied to any programming language. Some of the concepts were present in a much less developed form in our above-cited work; in retrospect we could have used the algebra to better formalize the semantics we were preserving in our dynamic scheduling runtime system. The runtime system (SOS) cited in that paper is incorporated in a version of PC (Parallel C, an extension of C described in [18]), and we are currently working on extensions to the language and runtime system based on the algebra. (Versions of this code are available from the authors).

Further work includes the possible application of the algebra of synchronization to optimization of synchronization code.

## A Alternate Semaphore Proof

The following proof of the first semaphore result (Theorem 4.1) does not rely on the algebra of synchronization we developed and is included to demonstrate not only the correctness of the first proof, but also its elegance and the related power of the theory. Recall that Theorem 4.1 states that the order of process release by a set of semaphores depends only on the P instructions, not on the V instructions.

**Proof:** We claim:

$P_1 \dots P_n \times V_i \dots V_j$  is the same (semantically) for any permutation of  $V_i$ .

This is true because any process  $Q$  that reaches  $x$  has blocked all other processes attempting it at some semaphore  $P_i$ .  $Q$  executing  $x$  is always free to continue and release all semaphores  $P_i$  in  $P_1 \dots P_n$ , and allow all held processes to continue.

The held processes may be released in different order depending on the permutation of the  $V_i$ . There are three possibilities:

1. All processes need to execute  $P_1 \dots P_n$  in that sequence to reach  $x$ . In this case, all processes will be held in the queue at  $P_1$ , because  $Q$  has blocked  $P_1$  and allowed no other process to get by. Held processes will be released in queue order, independent of the permutation of  $V_i$ .
2. Some processes execute a subsequence of the  $P_i$ . In that case, there may be a process  $R$  held at some  $P_k$ ,  $k > 1$ , while other processes  $T$  are held at  $P_1$ . Assume this is the case. Then, either  $P_1$  is released before  $P_k$ , or  $P_k$  is released before  $P_1$ . If  $P_k$  is released before  $P_1$ , then  $R$  will hold the lock at  $P_k$ , and all processes  $T$  will be held there. If  $P_1$  is released first, then the processes  $T$  will either reach  $P_k$  before it is released or after. If before, then they will be queued behind  $R$ . If after, then  $R$  holds the lock  $P_k$  and all processes  $T$  will be queued at  $R$ . In all cases,  $R$  executes  $X$  before any process in  $T$ . By induction extends to any set of processes  $R$  held at any of the  $P_i$ .

Therefore process execution order depends either on the queues or on the order of semaphores, and not on the permutation of  $V_i$ .

3. Some processes execute a permutation of the  $P_i$  to reach  $x$ .

Consider first processes  $S$  and  $R$ , such that  $S$  is held at  $P_s$  and  $R$  is held at  $P_r$ . Process  $Q$ , which is executing  $x$ , holds all locks.

$R$  must execute  $P_r$  before  $P_s$ , and  $S$  must execute  $P_s$  before  $P_r$  for this to happen, else each process would have been held at a different  $P_i$  (because  $Q$  holds all the locks and has therefore blocked everybody). In fact, it must be the case that  $P_r$  is the first  $P_i$  in the sequence  $R$  executes, and  $P_s$  is the first  $P_i$  executed by  $S$ .

Assume that  $S$  is released first. Now we have case 2, above: that is, the result is that  $R$  holds the lock at  $P_r$  and  $S$  is in the  $P_r$  queue. But in order to get here,  $S$  holds the  $P_s$  queue, which  $R$  will encounter after its release. Therefore we have deadlock because  $S$  is held in queue  $P_r$ ,  $Q$  is held in  $P_s$ , and neither gets to execute  $x$  and the  $V_i$  afterwards.

Same applies if R is released first, by symmetry. By induction, extends to N processes.

THEREFORE: The order of semaphore release does not affect the order of execution of the critical section by processes held in the semaphore => the permutation of Vi does not change the semantics of multiple semaphores.

**QED**

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