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Commutativity of Composition of some n-Dimensional Cellular Automata on Monoids

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Abstract

The local function of a cellular automaton with binary states can be expressed by a formula in propositional logic. If a local function is that of any reversible cellular automaton, its inverse function can also be expressed as a propositional logic formula, and using it as a local function, we can define the cellular automaton. The multiplication of these formulae in propositional logic yields the local function of the composition of two cellular automata.

In this paper, we consider logical formulae on a commutative monoid as the local functions of n-dimensional cellular automata. We discuss the commutativity of the multiplication of formulae and show some conditions for formulae to satisfy the commutativity of the composition of n-dimensional cellular automata.

Keywords: Cellular automata, monoid, propositional logic, composition, commutativity.

1 Introduction

Von Neumann and Ulam introduced cellular automata (CAs) as theoretical models capable of self-reproduction and universal computation [8]. Recently, CAs have been used to mean CAs on groups, and the mathematical theory of CAs has been developed in relation to the theory of groups [2]. Hedlund, Moore, and Drisko presented some algebraic properties of CAs on semigroups [3, 7]. Ishida et al. have introduced CAs on monoids using formulae in propositional logic, instead of local functions [6]. The definition and properties of CAs on monoids are quite different from those of CAs on groups mainly because of the absence of inverse elements for some elements in a monoid. The following analogy [5] exists between CAs and propositional logic.

CAs		Propositional logic
set of states $Q = \{0, 1\}$	$\Leftrightarrow$	set of truth values $Q = \{0, 1\}$
cell space G	$\Leftrightarrow$	set of propositional variables G
configuration $m \in Q^G$	$\Leftrightarrow$	valuation $m \in Q^G$
local function $f : Q^N \to Q$ (finite set $N \subseteq G$)	$\Leftrightarrow$	formula A

The composition of global transition functions and the multiplication of propositional logic are related to the reversibility of a given CA [4, 6]. The local function of a composed CA can be expressed as the multiplication of propositional logic. However, many properties of composed CAs on monoids have not been clarified.

In this paper, we examine logical formulae on commutative monoids as local functions of n-dimensional CAs on monoids. We discuss the commutativity of the multiplication of formulae and describe the conditions for formulae that can satisfy the commutativity of the composition of n-dimensional CAs on commutative monoids which is commutable for the composition with the CA of a local function using one cell in its neighborhood, for example, local function $f(x, y, z) = \neg x$ in elementary CAs. We provide examples of the composition that can satisfy commutativity for n-dimensional CAs on commutative monoids. Moreover, we present the conditions required to satisfy the commutativity of the composition of inverse functions of reversible CAs on monoids.

This paper is organized as follows. In Section 2, we review the fundamentals of propositional logic [1]. In Section 3, we introduce CAs on monoids using formulae instead of local functions [6], and present the formulae of some n-dimensional CAs. In Section 4, we introduce the multiplication of the formulae of CAs on monoids and describe the basic properties of the multiplication [6]. Moreover, we describe some properties of the composition of the formulae on commutative monoids. In Section 5, we discuss some conditions that can satisfy the commutativity of the composition of CAs on commutative monoids by the multiplication of formulae and provide examples of the compositions that can satisfy commutativity for n-dimensional CAs on commutative monoids. Furthermore, we present the conditions that can satisfy the commutativity of the composition of the inverse functions of CAs on monoids.

2 Propositional logic

First we review the fundamentals of propositional logic to draw an analogy between propositional logic [1] and the CAs theory [2].

Let X be a set of propositional variables, and $\perp$ and $\to$ be logical symbols. Formulae on X are defined by BNF:

$$A ::= x \mid \perp \mid A \to A \quad (x \in X)$$

Common abbreviations are used to introduce other logical symbols.

Negation:	$\neg A = A \to \perp,$
Verum:	$\top = \neg\perp,$
Disjunction:	$A \vee B = \neg A \to B,$
Conjunction:	$A \wedge B = \neg(A \to \neg B),$
Equivalence:	$A \leftrightarrow B = (A \to B) \wedge (B \to A),$
Exclusive or:	$A + B = \neg(A \leftrightarrow B).$

In what follows, we assume that $Q = (\{0, 1\}, \wedge, \vee, \neg)$ is a Boolean algebra of truth values. The implication operator $\Rightarrow$ on Q is defined by $p \Rightarrow q = \neg p \vee q$ for $p, q \in Q$. Operations $\Leftrightarrow$ (equivalence), $+$ (exclusive or (XOR), addition modulo 2) on Q are defined in the same way as the above abbreviations.

Definition 2.1. A *valuation* (interpretation) m for a set X is function $m : X \to Q$. For all formulae A on X, *truth value* $m[\![A]\!] \in Q$ of A with respect to m is inductively defined as follows:

- $m[\![x]\!] = m(x)$ for all propositional variables $x \in X$,
- $m[\![\perp]\!] = 0$,
- $m[\![A \to B]\!] = m[\![A]\!] \Rightarrow m[\![B]\!]$ for all formulae A and B on X. □

For two formulae A and B on X, we write $A \equiv B$ if $m[\![A]\!] = m[\![B]\!]$ for all valuations $m : X \to Q$. For the valuation m, the three properties $m[\![\neg A]\!] = \neg m[\![A]\!], m[\![a \vee B]\!] = m[\![A]\!] \vee b[\![B]\!], m[\![A \wedge B]\!] = m[\![A]\!] \wedge m[\![B]\!]$ hold from

$$m[\![\neg A]\!] = m[\![A \to \perp]\!] = m[\![A]\!] \Rightarrow m[\![\perp]\!] = \neg m[\![A]\!] \vee m[\![\perp]\!] = \neg m[\![A]\!],$$

$$\begin{aligned} m[\![A \vee B]\!] &= m[\![\neg A \to B]\!] \\ &= m[\![\neg A]\!] \Rightarrow m[\![B]\!] \\ &= \neg m[\![A]\!] \Rightarrow m[\![B]\!] \\ &= \neg\neg m[\![A]\!] \vee m[\![B]\!] \\ &= m[\![A]\!] \vee m[\![B]\!], \end{aligned}$$

$$\begin{aligned} m[\![A \wedge B]\!] &= m[\![\neg(A \to \neg B)]\!] \\ &= \neg m[\![A \to \neg B]\!] \\ &= \neg(m[\![A]\!] \Rightarrow \neg m[\![B]\!]) \\ &= \neg(\neg m[\![A]\!] \vee \neg m[\![B]\!]) \\ &= m[\![A]\!] \wedge m[\![B]\!]. \end{aligned}$$

Proposition 2.2. *Let A, B, C be formulae on X. Then, the following hold:*

1. $A \vee B \equiv B \vee A$, $(A \vee B) \vee C \equiv A \vee (B \vee C)$, $A \vee A \equiv A$, $A \vee \neg A \equiv \top$,
2. $A \wedge B \equiv B \wedge A$, $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$, $A \wedge A \equiv A$, $A \wedge \neg A \equiv \bot$,
3. $A + B \equiv B + A$, $(A + B) + C \equiv A + (B + C)$,
4. $\neg(\neg A) \equiv A$, $\neg(A \vee B) \equiv \neg A \wedge \neg B$, $\neg(A \wedge B) \equiv \neg A \vee \neg B$,
5. $A + A \equiv \bot$, $A + \bot \equiv A$, $A + \top \equiv \neg A$,
6. $(A \vee B) \wedge C \equiv (A \wedge C) \vee (B \wedge C)$, $(A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C)$,
7. $(A \wedge B) \vee (A \vee B) \equiv (A \vee B)$, $(A \wedge B) \wedge (A \vee B) \equiv (A \wedge B)$.

For the logical symbol $+$, there exists the following lemma:

Lemma 2.3. *Let A, B, C be formulae on X. Then, the following hold:*

1. $A + B \equiv (A \wedge \neg B) \vee (\neg A \wedge B)$,
2. $\neg(A + B) \equiv \neg A + B$,
3. $\neg A + \neg B \equiv A + B$.

Proof. Let A, B, C be formulae on X. By Definition 2.1 and Proposition 2.2,

$$\begin{aligned} A + B &\equiv \neg((A \to B) \wedge (B \to A)) \\ &\equiv \neg(A \to B) \vee \neg(B \to A) \\ &\equiv (A \wedge \neg B) \vee (\neg A \wedge B), \end{aligned}$$

$$\begin{aligned} \neg(A + B) &\equiv \neg((A \wedge \neg B) \vee (\neg A \wedge B)) \\ &\equiv (\neg A \vee B) \wedge (A \vee \neg B) \\ &\equiv (\neg A \wedge (A \vee \neg B)) \vee (B \wedge (A \vee \neg B)) \\ &\equiv ((\neg A \wedge A) \vee (\neg A \wedge \neg B)) \vee ((B \wedge A) \vee (B \wedge \neg B)) \\ &\equiv (\neg A \wedge \neg B) \vee (B \wedge A) \\ &\equiv \neg A + B (\equiv A + \neg B), \end{aligned}$$

$$\begin{aligned} \neg A + \neg B &\equiv (\neg A \wedge \neg(\neg B)) \vee (\neg(\neg A) \wedge \neg B) \\ &\equiv (\neg A \wedge B) \vee (A \wedge \neg B) \\ &\equiv A + B. \end{aligned}$$

□

By Lemma 2.3 and Proposition 2.2, the following proposition holds:

Proposition 2.4. *Let A, B, C be formulae on X. Then*

1. $A \wedge (B \vee C) \vee (B \wedge C) \equiv (A \vee (B \wedge C)) \wedge (B \vee C)$.
2. $A \wedge (B \vee C) \vee (B \wedge C) \equiv B \wedge (A \vee C) \vee (A \wedge C)$.
3. $A \wedge (A \vee B) \vee (A \wedge B) \equiv A$.
4. $\neg(A + B + C) \equiv \neg A + \neg B + \neg C$
5. $\neg A + \neg B + C \equiv A + B + C$
6. $\neg A + B + C \equiv \neg(A + B + C)$

Proof. Let A, B, C be formulae on X.

$$\begin{aligned} A \wedge (B \vee C) \vee (B \wedge C) &\equiv (A \vee (B \wedge C)) \wedge ((B \vee C) \vee (B \wedge C)) \\ &\equiv (A \vee (B \wedge C)) \wedge (B \vee C), \end{aligned}$$

$$\begin{aligned} A \wedge (B \vee C) \vee (B \wedge C) &\equiv ((A \wedge B) \vee (A \wedge C)) \vee (B \wedge C) \\ &\equiv ((B \wedge A) \vee (B \wedge C)) \vee (A \wedge C) \\ &\equiv B \wedge (A \vee C) \vee (A \wedge C). \end{aligned}$$

$$\begin{aligned} A \wedge (A \vee B) \vee (A \wedge B) &\equiv (A \vee (A \wedge B)) \wedge ((A \vee B) \vee (A \wedge B)) \\ &\equiv A \wedge (A \vee B) \\ &\equiv A \end{aligned}$$

$$\begin{aligned} \neg(A + B + C) &\equiv \neg(A) + (B + C) \\ &\equiv \neg A + \neg B + \neg C. \end{aligned}$$

$$\begin{aligned} \neg A + \neg B + C &\equiv (\neg A + \neg B) + C \\ &\equiv A + B + C. \end{aligned}$$

$$\begin{aligned} \neg A + B + C &\equiv \neg(A + B) + C \\ &\equiv \neg(A + B + C). \end{aligned}$$

□

3 CA on monoids

In this section, we describe CAs on monoids [6]. Let cell space $M = \{x^a | a \in \mathbb{N}^n\}$ $(n \in \mathbb{N})$ be a monoid with unit element e, and a neighborhood $N = \{x^{a_1}, \ldots, x^{a_m}\} \subset M$ $(a_1, \ldots, a_m \in \mathbb{N}^n)$. In what follows, we assume that the set of all formulae on M is denoted by $F(M)$.

Definition 3.1. For a formula A on M and $x^a \in M$, the *shifted formula* $x^a A$ on M is defined by induction on A :

1. $x^a x^b \in M$ (monoid multiplication in M) for all $x^b \in M$,
2. $x^a \bot \equiv \bot$,
3. $x^a(A \to B) \equiv x^a A \to x^a B$ for all $A, B \in F(M)$. □

The following states basic properties of the shifted formulae.

Proposition 3.2 ([6]). *Let A and B be formulae on M and $x^a, x^b \in M$. Then, the following hold:*

1. $eA \equiv A$,
2. $(x^a x^b)A \equiv x^a(x^b A)$,
3. $x^a(\neg A) \equiv \neg(x^a A)$,
4. $x^a(A \vee B) \equiv x^a A \vee x^a B$,
5. $x^a(A \wedge B) \equiv x^a A \wedge x^a B$,
6. $x^a(A + B) \equiv x^a A + x^a B$. □

Generally, $xy \neq yx$ for a monoid M and elements x, y of M.

A function (valuation) $m : M \to Q$ is called *configuration* on M in the context of CA. We denote by Q^M the set of all configurations $m : M \to Q$. For $q \in Q$ the constant configuration $\hat{q} \in Q^M$ is defined by $\hat{q}(x^a) = q$ for all $x^a \in M$.

Definition 3.3. For a finite subset $N \subseteq M$ and a formula $A \in F(N)$, we define a function $T_A : Q^M \to Q^M$ by $T_A(m)(x^a) = m[\![x^a A]\!]$ for all $m \in Q^M$ and $x^a \in M$. Function T_A is called the *global transition function* (or CA), defined by A on M, and A is called the *local formula.* □

Since the local function of a CA with binary states is a Boolean function with finite variables, it has a disjunctive normal form. We present some examples of the disjunctive normal form of the local function of a CA with binary states.

Consider $\mathbb{N}$ as the additive monoid of all natural numbers. $\mathbb{N}^n$ is the additive monoid. In the example, $M = \{x^y | y \in \mathbb{N}^n\}$, $N = \{x^a, x^b, x^c, x^d\} \subseteq M$ $(a, b, c, d \in \mathbb{N}^n)$. Let the local function $f : Q^N \to Q$ be defined according to the following table:

$x^a x^b x^c x^d$	1111	1110	1101	1100	1011	1010	1001	1000
f	0	0	1	1	1	1	1	1
$x^a x^b x^c x^d$	0111	0110	0101	0100	0011	0010	0001	0000
f	0	0	0	0	0	0	1	1

The disjunctive normal form A of f is as follows:

$$\begin{aligned} A \quad \equiv \quad & (x^a \wedge x^b \wedge \neg x^c \wedge x^d) \vee (x^a \wedge x^b \wedge \neg x^c \wedge \neg x^d) \vee (x^a \wedge \neg x^b \wedge x^c \wedge x^d) \\ & \vee (x^a \wedge \neg x^b \wedge x^c \wedge \neg x^d) \vee (x^a \wedge \neg x^b \wedge \neg x^c \wedge x^d) \vee (x^a \wedge \neg x^b \wedge \neg x^c \wedge \neg x^d) \\ & \vee (\neg x^a \wedge \neg x^b \wedge \neg x^c \wedge x^d) \vee (\neg x^a \wedge \neg x^b \wedge \neg x^c \wedge \neg x^d). \end{aligned}$$

The simplified formula for A is as follows.

$$A \equiv x^a \wedge (\neg x^b \vee \neg x^c) \vee (\neg x^b \wedge \neg x^c)$$

The class of the CAs defined by logical formulae contains the class of the binary CAs defined by local functions.

Proposition 3.4 ([6]). *Let $x^a, x^b \in M, m \in Q^M$ and $A, B \in F(M)$. Then the following hold:*

1. $T_{x^b}(m)(x^a) = m[\![x^a x^b]\!]$. *In particular, T_e is the identity function on Q^M.*
2. $T_\perp(m) = \hat{0}$,
3. $T_{A \to B}(m)(x^a) = T_A(m)(x^a) \Rightarrow T_B(m)(x^a)$,
4. $T_{\neg A}(m)(x^a) = \neg(T_A(m)(x^a))$,

5. $T_{A\vee B}(m)(x^a) = T_A(m)(x^a) \vee T_B(m)(x^a)$,
6. $T_{A\wedge B}(m)(x^a) = T_A(m)(x^a) \wedge T_B(m)(x^a)$,
7. $T_{A+B}(m)(x^a) = T_A(m)(x^a) + T_B(m)(x^a)$,

We define shifted configurations using the monoid action.

Definition 3.5. For a configuration $m \in Q^M$ and $x^a \in M$, the *shifted configuration* $(x^a)^\circ m \in Q^M$ is defined as $((x^a)^\circ m)(x^b) = m[\![x^a x^b]\!]$ for all $x^b \in M$. □

We show an example of the shifted configuration. Let $\mathbb{N}^2$ be an additive monoid and the cell space $M' = \{x^y | y \in \mathbb{N}^2\}$ be a commutative monoid. For $x^{<a,b>}, x^{<c,d>} \in M'$, we can calculate $((x^{<a,b>})^\circ m)(x^{<c,d>})$ as follows;

$$((x^{<a,b>})^\circ m)(x^{<c,d>}) = m[\![x^{<a,b>} x^{<c,d>}]\!] = m[\![x^{<a+c,b+d>}]\!].$$

The following states the basic properties of the shifted configurations.

Proposition 3.6 ([6])**.** *Let $x^a, x^b \in M, m \in Q^M$, and $A, B \in F(M)$. Then the following hold:*

1. $e^\circ m = m$,
2. $(x^a x^b)^\circ m = (x^b)^\circ((x^a)^\circ m)$,
3. $((x^a)^\circ m)[\![A]\!] = m[\![x^a A]\!]$,
4. $T_A(((x^a)^\circ m) = (x^a)^\circ(T_A(m))$,
5. $A \equiv B$ *implies* $(x^a)A \equiv (x^a)B$. □

In Table 1, we provide examples of the formulae of the local functions for n-dimensional CAs. Let the cell space $M = \{x^y | y \in \mathbb{N}^n\}$ $(n \in \mathbb{N})$ be a commutative monoid, and a neighborhood $N = \{x^{a_1}, \ldots, x^{a_k}\} \subset M$ $(a_1, \ldots a_k \in \mathbb{N}^n)$. Let $x^a, x^b, x^c \in N$, and $a \neq b, b \neq c, a \neq c$.

Table 1: Formulae of local functions

Type-1	x^a
Type-2	$\neg x^a$
Type-3	$(x^a \wedge (x^b \vee x^c)) \vee (x^b \wedge x^c)$
Type-4	$(\neg x^a \wedge (x^b \vee x^c)) \vee (x^b \wedge x^c)$
Type-5	$\neg((\neg x^a \wedge (x^b \vee x^c)) \vee (x^b \wedge x^c))$
Type-6	$\neg((x^a \wedge (x^b \vee x^c)) \vee (x^b \wedge x^c))$
Type-7	$x^a + x^b + x^c$
Type-8	$\neg(x^a + x^b + x^c)$

For the formulae of Type-3 and Type-6, the following equations hold

$$(x^a \wedge (x^b \vee x^c)) \vee (x^b \wedge x^c) \equiv (x^b \wedge (x^a \vee x^c)) \vee (x^a \wedge x^c) \equiv (x^c \wedge (x^a \vee x^b)) \vee (x^a \wedge x^b),$$
$$\neg((x^a \wedge (x^b \vee x^c)) \vee (x^b \wedge x^c)) \equiv \neg((x^b \wedge (x^a \vee x^c)) \vee (x^a \wedge x^c)) \equiv \neg((x^c \wedge (x^a \vee x^b)) \vee (x^a \wedge x^b))$$

by Proposition 2.2 and 2.4.

The CAs over the cell space $M = \{x^n | n \in \mathbb{N}\}$ and the neighborhood $N = \{x^0(= e), x^1, x^2\} \subset M$ are called elementary CAs. In Table 2, we provide examples of the formulae of the local function f'_l for elementary CAs. In the following, the local function f'_l of a number l is the local function satisfying the following equation:

$$l = \sum_{m \in Q^N} f'_l(m) \cdot 2^{(2^2 m[\![x^2]\!] + 2m[\![x^1]\!] + m[\![x^0]\!])}.$$

Table 2: Examples of the local functions of each type of elementary CA

Type-1	$f'_{170}, f'_{204}, f'_{240}$
Type-2	$f'_{15}, f'_{51}, f'_{85}$
Type-3	f'_{232}
Type-4	$f'_{142}, f'_{178}, f'_{212}$
Type-5	$f'_{43}, f'_{77}, f'_{113}$
Type-6	f'_{23}
Type-7	f'_{150}
Type-8	f'_{105}

Let the cell space $M' = \{x^y | y \in \mathbb{N}^2\}$ be a commutative monoid and the neighborhood be $N' = \{x^{<0,0>}, x^{<0,1>}, x^{<1,0>}, x^{<1,1>}\} \subset M'$. The local function f_k of a number k is the local function satisfying the following equation:

$$k = \sum_{m \in Q^{N'}} f_k(m) \cdot 2^{(2^3 m[\![x^{<1,1>}]\!] + 2^2 m[\![x^{<1,0>}]\!] + 2^1 m[\![x^{<0,1>}]\!] + 2^0 m[\![x^{<0,0>}]\!])}.$$

For example, we calculate one of the local functions for the formula $A = (x^{<1,1>} \wedge (x^{<1,0>} \vee x^{<0,1>})) \vee (x^{<1,0>} \wedge x^{<0,1>})$ of Type-3. In the case of m such that $m[\![x^{<1,1>}]\!] = 1, m[\![x^{<1,0>}]\!] = 0, m[\![x^{<0,1>}]\!] = 1$, and $m[\![x^{<0,0>}]\!] = 1$, then

$$\begin{aligned} m[\![A]\!] &= (m[\![x^{<1,1>}]\!] \wedge (m[\![x^{<1,0>}]\!] \vee m[\![x^{<0,1>}]\!])) \vee (m[\![x^{<1,0>}]\!] \wedge m[\![x^{<0,1>}]\!]) \\ &= (1 \wedge (0 \vee 1)) \vee (0 \wedge 1) \\ &= 1 \end{aligned}$$

By calculating the value $m[\![A]\!]$ for all the other cases of $m \in Q^{N'}$, the local function f for A is described by the following table;

$x^{<1,1>}x^{<1,0>}x^{<0,1>}x^{<0,0>}$	1111	1110	1101	1100	1011	1010	1001	1000
$m[\![A]\!]$	1	1	1	1	1	1	0	0
$x^{<1,1>}x^{<1,0>}x^{<0,1>}x^{<0,0>}$	0111	0110	0101	0100	0011	0010	0001	0000
$m[\![A]\!]$	1	1	0	0	0	0	0	0

The number of f for A is 64704 from

$$1 \times 2^{2^3 \times 1 + 2^2 \times 1 + 2^1 \times 1 + 2^0 \times 1} + 1 \times 2^{2^3 \times 1 + 2^2 \times 1 + 2^1 \times 1 + 2^0 \times 0} + \cdots + 0 \times 2^{2^3 \times 0 + 2^2 \times 0 + 2^1 \times 0 + 2^0 \times 0} = 64704.$$

Thus the local function for A is f_{64704} and also we have that the local functions for the formulae $(x^{<1,0>} \wedge (x^{<1,1>} \vee x^{<0,1>})) \vee (x^{<1,1>} \wedge x^{<0,1>})$ and $(x^{<0,1>} \wedge (x^{<1,1>} \vee x^{<1,0>})) \vee (x^{<1,1>} \wedge x^{<1,0>})$ are the same f_{64704}. By calculating the local functions of other formulae of Type-3 we can show that there are four local functions $f_{64704}, f_{64160}, f_{61064}, f_{59624}$ of Type-3. In Table 3, we provide examples of the local function of each type for 2-dimensional CA over a commutative monoid M' and a neighborhood N'.

4 Multiplication of formulae

In this section, we define the multiplication of the formulae of CA on monoids [6] and its properties.

Definition 4.1. Let A and C be formulae on M and $x^a \in M$. The multiplication $A \circ C$ of A and C is inductively defined as follows:

1. $x^a \circ C \equiv x^a C$ (shifted formula) is already defined for $x^a \in M$,

Table 3: Examples of local functions of each type

Type-1	$f_{43690}, f_{52428}, f_{61680}, f_{65280}$
Type-2	$f_{255}, f_{3855}, f_{13107}, f_{21845}$
Type-3	$f_{59624}, f_{61064}, f_{64160}, f_{64704}$
Type-4	$f_{36494}, f_{35054}, f_{41210}, f_{49404}, f_{45746}, f_{47906}, f_{44810}, f_{53004}, f_{54484}, f_{56644}, f_{62800}, f_{62256}$
Type-5	$f_{3279}, f_{12531}, f_{2735}, f_{20725}, f_{8891}, f_{17629}, f_{11051}, f_{19789}, f_{16131}, f_{24325}, f_{30481}, f_{29041}$
Type-6	$f_{831}, f_{1375}, f_{4471}, f_{5911}$
Type-7	$f_{38550}, f_{39270}, f_{42330}, f_{49980}$
Type-8	$f_{15555}, f_{23205}, f_{26265}, f_{26985}$

2. $\bot \circ C \equiv \bot$,
3. $(A \to B) \circ C \equiv A \circ C \to B \circ C$ for formulae A, B on M. □

The following states the basic properties of the multiplication of the formulae.

Proposition 4.2 ([6]). *Let $A, B, C \in F(M)$ and $x^a, x^b \in M$. Then the following hold:*

1. $A \circ e \equiv A$, $A \circ (x^a x^b) \equiv (A \circ x^a) \circ x^b$,
2. *Either* $A \circ \bot \equiv \bot$ *or* $A \circ \bot \equiv \top$,
3. $(\neg A) \circ B \equiv \neg(A \circ B)$,
4. $(A \vee B) \circ C \equiv A \circ C \vee B \circ C$,
5. $(A \wedge B) \circ C \equiv A \circ C \wedge B \circ C$,
6. $(A + B) \circ C \equiv A \circ C + B \circ C$,
7. $(A \circ B) \circ C \equiv A \circ (B \circ C)$,

Proof. Let $A, B, C \in F(M)$ and $x^a, x^b \in M$. These propositions are proved by Definition 3.1 and 4.1 and structural induction on A and B.
1. $A \circ e \equiv A$:

$$\begin{aligned} x^a \circ e &\equiv x^a, \quad \{\, e : \text{unit},\ x^a \in M \,\} \\ \bot \circ e &\equiv \bot. \end{aligned}$$

We assume $A \circ e \equiv A$ and $B \circ e \equiv B$, then

$$\begin{aligned} (A \to B) \circ e &\equiv A \circ e \to B \circ e \\ &\equiv A \to B. \end{aligned}$$

$A \circ (x^a x^b) \equiv (A \circ x^a) \circ x^b$:

$$\begin{aligned} x^c \circ (x^a x^b) &\equiv (x^c \circ x^a) \circ x^b, \\ \bot \circ (x^a x^b) &\equiv \bot \\ &\equiv (\bot \circ x^a) \circ x^b. \end{aligned}$$

We assume $A \circ (x^a x^b) \equiv (A \circ x^a) \circ x^b$ and $B \circ (x^a x^b) \equiv (B \circ x^a) \circ x^b$, then

$$\begin{aligned} (A \to B) \circ (x^a x^b) &\equiv A \circ (x^a x^b) \to B \circ (x^a x^b) \\ &\equiv (A \circ x^a) \circ x^b \to (B \circ x^a) \circ x^b \\ &\equiv (A \circ x^a \to B \circ x^a) \circ x^b \\ &\equiv ((A \to B) \circ x^a) \circ x^b. \end{aligned}$$

2. Either $A \circ \bot \equiv \bot$ or $A \circ \bot \equiv \top$:

$$\begin{aligned} x^a \circ \bot &\equiv \bot, \quad \{\ x^a \in M\ \} \\ \bot \circ \bot &\equiv \bot. \end{aligned}$$

We assume that either $A \circ \bot \equiv \bot$ or $A \circ \bot \equiv \top$, and either $B \circ \bot \equiv \bot$ or $B \circ \bot \equiv \top$. Then

$$\begin{aligned} (A \to B) \circ \bot &\equiv A \circ \bot \to B \circ \bot \\ &\equiv \bot \text{ or } \top. \end{aligned}$$

3. $(\neg A) \circ B \equiv \neg(A \circ B)$:

$$\begin{aligned} (\neg A) \circ B &\equiv (A \to \bot) \circ B \\ &\equiv A \circ B \to \bot \circ B \\ &\equiv A \circ B \to \bot \\ &\equiv \neg(A \circ B). \end{aligned}$$

4. $(A \vee B) \circ C \equiv A \circ C \vee B \circ C$: By $(\neg A) \circ C \equiv \neg(A \circ C)$,

$$\begin{aligned} (A \vee B) \circ C &\equiv (\neg A \to B) \circ C \\ &\equiv (\neg A) \circ C \to B \circ C \\ &\equiv \neg(A \circ C) \to B \circ C \\ &\equiv A \circ C \vee B \circ C \end{aligned}$$

5. $(A \wedge B) \circ C \equiv A \circ C \wedge B \circ C$: By $(\neg B) \circ C \equiv \neg(B \circ C)$,

$$\begin{aligned} (A \wedge B) \circ C &\equiv (\neg(A \to \neg B)) \circ C \\ &\equiv \neg((A \to \neg B) \circ C) \\ &\equiv \neg(A \circ C \to \neg(B \circ C)) \\ &\equiv A \circ C \wedge B \circ C. \end{aligned}$$

6. $(A + B) \circ C \equiv A \circ C + B \circ C$:

$$\begin{aligned} (A + B) \circ C &\equiv ((A \leftrightarrow B) \to \bot) \circ C \\ &\equiv (A \circ C \leftrightarrow B \circ C) \to \bot \circ C \\ &\equiv (A \circ C \leftrightarrow B \circ C) \to \bot \\ &\equiv A \circ C + B \circ C. \end{aligned}$$

7. $(A \circ B) \circ C \equiv A \circ (B \circ C)$:

$$\begin{aligned} (x^a \circ B) \circ C &\equiv x^a \circ (B \circ C), \\ (\bot \circ B) \circ C &\equiv \bot \circ C \equiv \bot \equiv \bot \circ (B \circ C), \end{aligned}$$

The equation $(x^a \circ B) \circ C \equiv x^a \circ (B \circ C)$ is proved by structural induction on B as follows;

$$\begin{aligned} (x^a \circ x^b) \circ C &\equiv x^a \circ (x^b \circ C), \\ (x^a \circ \bot) \circ C &\equiv \bot \circ C \equiv \bot \equiv x^a \circ (\bot \circ C), \\ (x^a \circ (B \to B')) \circ C &\equiv (x^a B \to x^a B') \circ C \\ &\equiv (x^a B) \circ C \to (x^a B') \circ C \\ &\equiv x^a \circ (B \circ C) \to x^a \circ (B' \circ C) \\ &\equiv x^a \circ (B \circ C \to B' \circ C) \\ &\equiv x^a \circ ((B \to B') \circ C). \end{aligned}$$

Assume that $(A \circ B) \circ C \equiv A \circ (B \circ C)$ and $(A' \circ B) \circ C \equiv A' \circ (B \circ C)$ hold. Then

$$\begin{aligned} ((A \to A') \circ B) \circ C &\equiv (A \circ B \to A' \circ B) \circ C \\ &\equiv (A \circ B) \circ C \to (A' \circ B) \circ C \\ &\equiv A \circ (B \circ C) \to A' \circ (B \circ C) \\ &\equiv (A \to A') \circ (B \circ C). \end{aligned}$$

□

In general, $A \circ B \equiv B \circ A, (\neg A) \circ B \equiv A \circ (\neg B)$, $A \circ (B \to C) \equiv A \circ B \to A \circ C$ and $A \circ (B \vee C) \equiv A \circ B \vee A \circ C$ do not need to hold. (For example, $\neg x^a \circ (x^b \vee x^c) \equiv \neg x^a x^b \wedge \neg x^a x^c, (x^b \vee x^c) \circ (\neg x^a) \equiv \neg x^a x^b \vee \neg x^a x^c$ $(x^a, x^b, x^c \in M)$.)

Definition 4.3. For a formula $A \in F(M)$, the set $V(A)$ of all variables in A is defined as follows:

1. $V(x^a) = \{x^a\}$ for all $x^a \in M$,
2. $V(\bot) = \emptyset$,
3. $V(B \to C) = V(B) \cup V(C)$ for all $B, C \in F(M)$. □

The set $V(A)$ is a finite subset of M and serves as the neighborhood of the local formula A. It is verified that $m|_{V(A)} = m'|_{V(A)}$ implies $m[\![A]\!] = m'[\![A]\!]$ and that $V(A \circ B) = V(A)V(B)$ holds, where $V(A)V(B) = \{x^a x^b \in M \mid x^a \in V(A) \wedge x^b \in V(B)\}$.

Proposition 4.4 ([6]). *Let $A, B \in F(M)$ and $x^a \in M$. Then the following hold:*

1. *If $A \equiv x^a$, then $x^a \in V(A)$.*
2. *If $A \circ B \equiv e$, then there exist $x^a \in V(A)$ and $x^b \in V(B)$ so that $x^a x^b = e$.*

The composition $S \circ T$ of a function $T : Q^M \to Q^M$, followed by a function $S : Q^M \to Q^M$, is defined as usual:

$$\forall m \in Q^M.\ (S \circ T)(m) = S(T(m)).$$

Although the composition of the transition functions of CAs seems to behave awkwardly in terms of traditional local functions $f : Q^N \to Q$, the multiplication of formulae directly represents the composition of global transition functions.

Theorem 4.5 ([5]). *For all formulae $A, C \in F(M)$, $T_A \circ T_C = T_{A \circ C}$.*

Definition 4.6. The transition function $T_A : Q^M \to Q^M$ defined by formula A is *reversible* if it is bijective, with $T_A^{-1} = T_B$ for some formula B. Formula A is *reversible* if there exists some formula B on M so that $B \circ A \equiv e$ and $A \circ B \equiv e$.

By the virtue of Theorem 4.5 Proposition 4.5 and Proposition 3.4 (8), A is reversible iff T_A is reversible.

Consider the two reversible CAs with respective formulae A and B on monoid M. There are formulae $A^{-1}, B^{-1} \in F(M)$ that satisfy $A \circ A^{-1} \equiv e, B \circ B^{-1} \equiv e$. Then, $(A \circ B) \circ (B^{-1} \circ A^{-1}) \equiv e$. Therefore, $T_{A \circ B}$ is reversible, and $T_{A \circ B}^{-1} = T_{B^{-1} \circ A^{-1}}$.

Proposition 4.7 ([6]). *Let $A, B, C \in F(M)$. Then, the following hold:*

1. *$T_A(m)[\![B]\!] = m[\![B \circ A]\!]$,*
2. *$A \equiv A'$ and $B \equiv B'$ imply $A \circ B \equiv A' \circ B'$,*
3. *If $A \circ B \equiv e$ and $C \circ A \equiv e$, then $B \equiv C$.*
4. *A and B are reversible iff so are $A \circ B$ and $B \circ A$.*

Proof. Let $A, B \in F(M)$.
1. $T_A(m)[\![B]\!] = m[\![B \circ A]\!]$: By Proposition 3.2, Definition 3.3 and Theorem 4.5,

$$\begin{aligned} m[\![B \circ A]\!] &= m[\![e(B \circ A)]\!] \\ &= T_{B \circ A}(m)(e) \\ &= (T_B \circ T_A)(m)(e) \\ &= (T_B(T_A(m))(e) \\ &= T_A(m)[\![eB]\!] \\ &= T_A(m)[\![B]\!]. \end{aligned}$$

2. $A \equiv A'$ and $B \equiv B'$ imply $A \circ B \equiv A' \circ B'$:
Assume $A \equiv A'$ and $B \equiv B'$. Then

$$\begin{aligned} m[\![A \circ B]\!] &= T_B(m)[\![A]\!] \\ &= T_{B'}(m)[\![A]\!] \\ &= T_{B'}(m)[\![A']\!] \\ &= m[\![A' \circ B']\!]. \end{aligned}$$

3. By Proposition 4.2,

$$B \equiv (C \circ A) \circ B \equiv C \circ (A \circ B) \equiv C.$$

4. Consider the two reversible CAs with respective formulae A and B on monoid M. There are formulae $A^{-1}, B^{-1} \in F(M)$ that satisfy $A \circ A^{-1} \equiv e, B \circ B^{-1} \equiv e$. By Proposition 4.2,

$$(A \circ B) \circ (B^{-1} \circ A^{-1}) \equiv (A \circ (B \circ B^{-1}) \circ A^{-1}) \equiv e,$$
$$(B \circ A) \circ (A^{-1} \circ B^{-1}) \equiv (B \circ (A \circ A^{-1}) \circ B^{-1}) \equiv e.$$

Therefore, $A \circ B$ and $B \circ A$ are reversible.

Consider the two reversible CAs with respective formulae $A \circ B, B \circ A$ on monoid M. There are formulae $(A \circ B)^{-1}, (B \circ A)^{-1} \in F(M)$ that satisfy $(A \circ B) \circ (A \circ B)^{-1} \equiv e, (B \circ A) \circ (B \circ A)^{-1} \equiv e$. By Proposition 4.2,

$$(A \circ B) \circ (A \circ B)^{-1} \equiv A \circ (B \circ (A \circ B)^{-1}) \equiv e,$$
$$(B \circ A) \circ (B \circ A)^{-1} \equiv B \circ (A \circ (B \circ A)^{-1}) \equiv e.$$

Therefore, A and B are reversible. □

If all elements x, y of M satisfy $xy = yx$, M is called a commutative monoid. An element x of M is *right cancellable* if $yx = zx$ implies $y = z$ for all $y, z \in M$. An element x of M is *left cancellable* if $xy = xz$ implies $y = z$ for all $y, z \in M$. If all elements x of M is *right cancellable* and *left cancellable*, then M is called a cancellable monoid.

Let $\mathbb{N}$ with operator $+$ be an additive monoid , $n \in \mathbb{N}$ and $M = \{x^m | m \in \mathbb{N}^n\}$ satisfy

$$\forall x^{<a_1,a_2,\ldots,a_k>}, x^{<b_1,b_2,\ldots,b_k>} \in M, x^{<a_1,a_2,\ldots,a_k>} x^{<b_1,b_2,\ldots,b_k>} = x^{<a_1+b_1,a_2+b_2,\ldots,a_k+b_k>}.$$

Then, M is a cancellable monoid and a commutative monoid.

Lemma 4.8. *Let M be a commutative monoid and $x^a, x^b, x^c \in M$. Then the following hold:*

1. $x^a \circ x^c \equiv x^c \circ x^a$
2. $\bot \circ x^c \equiv x^c \circ \bot$
3. $(x^a \to x^b) \circ x^c \equiv x^c \circ (x^a \to x^b)$
4. $(x^a \vee x^b) \circ x^c \equiv x^c \circ (x^a \vee x^b)$
5. $(x^a \wedge x^b) \circ x^c \equiv x^c \circ (x^a \wedge x^b)$
6. $(x^a + x^b) \circ x^c \equiv x^c \circ (x^a + x^b)$

Proof. Let M be a commutative monoid and $x^a, x^b, x^c \in M$. By Definition 3.1, 4.1, and the commutative monoid action,

$$x^a \circ x^c \equiv x^a x^c \equiv x^c x^a \equiv x^c \circ x^a,$$
$$\bot \circ x^c \equiv \bot x^c \equiv x^c \bot \equiv x^c \circ \bot,$$

$$\begin{aligned} (x^a \vee x^b) \circ x^c &\equiv x^a x^c \vee x^b x^c \\ &\equiv x^c x^a \vee x^c x^b \\ &\equiv x^c (x^a \vee x^b) \\ &\equiv x^c \circ (x^a \vee x^b). \end{aligned}$$

The proofs of the other formulae are similar and omitted. □

Lemma 4.8 shows that if M is a commutative monoid, $A \in F(M)$, and $x^a \in M$. Then, $x^a \circ A \equiv A \circ x^a$.

Lemma 4.9. *Let M be a commutative monoid and $x^a, x^b \in M$. Then the following hold:*

1. $x^a \circ (\neg x^b) \equiv (\neg x^b) \circ x^a$
2. $(\neg x^a) \circ (\neg x^b) \equiv (\neg x^b) \circ (\neg x^a) \equiv x^a \circ x^b$

Proof. Let M be a commutative monoid and $x^a, x^b \in M$. By Definition 3.1, 4.1, and the commutative monoid action,

$$x^a \circ (\neg x^b) \equiv x^a(\neg x^b) \equiv \neg(x^a x^b) \equiv \neg x^b x^a \equiv (\neg x^b) \circ x^a,$$

$$\begin{aligned}
(\neg x^a) \circ (\neg x^b) &\equiv \neg(x^a \circ (\neg x^b)) \\
&\equiv \neg(\neg x^a x^b) \\
&\equiv x^a x^b \\
&\equiv x^b x^a \\
&\equiv \neg(\neg x^b x^a) \\
&\equiv \neg(x^b \circ (\neg x^a)) \\
&\equiv (\neg x^b) \circ (\neg x^a).
\end{aligned}$$

□

Proposition 4.10. *Let M be a commutative monoid, and $x^a \in M$, $A, B, C \in M \cup \{\neg x^b | x^b \in M\}$. Then, the following hold:*

1. $\neg x^a \circ (A \wedge (B \vee C) \vee (B \wedge C)) \equiv (A \wedge (B \vee C) \vee (B \wedge C)) \circ \neg x^a$
2. $\neg x^a \circ \neg(A \wedge (B \vee C) \vee (B \wedge C))$
   $\equiv \neg(A \wedge (B \vee C) \vee (B \wedge C)) \circ \neg x^a$
3. $\neg x^a \circ (A + B + C) \equiv (A + B + C) \circ \neg x^a$
4. $\neg x^a \circ \neg(A + B + C) \equiv \neg(A + B + C) \circ \neg x^a$

Proof. Let M be a commutative monoid, $x^a \in M$, and $A, B, C \in M \cup \{\neg x^b | x^b \in M\}$.

$$\begin{aligned}
(A \wedge (B \vee C) \vee (B \wedge C)) \circ \neg x^a &\equiv (A \circ (\neg x^a) \wedge (B \circ (\neg x^a) \vee C \circ (\neg x^a)) \\
&\quad \vee (B \circ (\neg x^a) \wedge C \circ (\neg x^a))) \\
&\equiv \neg x^a A \wedge (\neg x^a B \vee \neg x^a C) \vee (\neg x^a B \wedge \neg x^a C) \\
&\equiv \neg x^a A \vee (\neg x^a B \wedge \neg x^a C) \wedge (\neg x^a B \vee \neg x^a C) \\
&\equiv \neg(x^a A \wedge (x^a B \vee x^a C) \vee (x^a B \wedge x^a C) \\
&\equiv \neg(x^a(A \wedge (B \vee C) \vee (B \wedge C))) \\
&\equiv \neg x^a \circ (A \wedge (B \vee C) \vee (B \wedge C)),
\end{aligned}$$

$$\begin{aligned}
\neg(A \wedge (B \vee C) \vee (B \wedge C)) \circ \neg x^a &\equiv \neg(A \wedge (B \vee C) \vee (B \wedge C) \circ \neg x^a) \\
&\equiv \neg(A \circ (\neg x^a) \wedge (B \circ (\neg x^a) \vee C \circ (\neg x^a)) \\
&\quad \vee (B \circ (\neg x^a) \wedge C \circ (\neg x^a))) \\
&\equiv \neg(\neg x^a A \wedge (\neg x^a B \vee \neg x^a C) \vee (\neg x^a B \wedge \neg x^a C)) \\
&\equiv (x^a A \vee (x^a B \wedge x^a C)) \wedge (\neg x^a B \vee \neg x^a C) \\
&\equiv x^a A \wedge (x^a B \vee x^a C) \vee (\neg x^a B \wedge \neg x^a C) \\
&\equiv \neg(x^a A \wedge (x^a B \vee x^a C) \vee (x^a B \wedge x^a C) \\
&\equiv \neg(\neg(x^a(A \wedge (B \vee C) \vee (B \wedge C)))) \\
&\equiv \neg(x^a \circ \neg((A \wedge (B \vee C) \vee (B \wedge C)))) \\
&\equiv \neg x^a \circ \neg(A \wedge (B \vee C) \vee (B \wedge C)),
\end{aligned}$$

$$\begin{aligned}
(A+B+C)\circ\neg x^a &\equiv (A\circ(\neg x^a)+B\circ(\neg x^a)+C\circ(\neg x^a))\\
&\equiv \neg Ax^a+\neg Bx^a+\neg Cx^a\\
&\equiv \neg x^aA+\neg x^aB+\neg x^aC\\
&\equiv \neg x^aA+\neg x^aB+\neg x^aC\\
&\equiv \neg(x^aA+x^aB+x^aC)\\
&\equiv \neg(x^a(A+B+C))\\
&\equiv \neg x^a\circ(A+B+C),
\end{aligned}$$

$$\begin{aligned}
\neg(A+B+C)\circ\neg x^a &\equiv \neg(A\circ(\neg x^a)+B\circ(\neg x^a)+C\circ(\neg x^a))\\
&\equiv \neg(\neg Ax^a+\neg Bx^a+\neg Cx^a)\\
&\equiv Ax^a+Bx^a+Cx^a\\
&\equiv x^aA+x^aB+x^aC\\
&\equiv \neg\neg(x^aA+x^aB+x^aC)\\
&\equiv \neg(x^a\circ\neg(A+B+C))\\
&\equiv \neg x^a\circ\neg(A+B+C),
\end{aligned}$$

by Proposition 2.4, Lemma 4.8 and Lemma 4.9. □

5 Commutativity of composition of CAs

In this section, we show the commutativity of the compositions of non elementary CAs on commutative monoids. Consider $\mathbb{N}$ as the additive monoid of all natural numbers and $\mathbb{N}^n$ as an additive monoid ($n \in \mathbb{N}$). We assume $M = \{x^a | a \in \mathbb{N}^n\}$, and a neighborhood $N = \{x^{a_1}, x^{a_2}, \ldots, x^{a_k}\}$ is the subset of M. Let $x^{a_i} = x^{<b_1,b_2,\ldots,b_n>}, x^{a_j} = x^{<c_1,c_2,\ldots,c_n>}$ ($a_i, a_j \in \mathbb{N}^n$). The operator of M is defined by

$$\forall x^{a_i}, x^{a_j} \in M, x^{a_i}x^{a_j} = x^{<b_1+c_1,b_2+c_2,\ldots,b_n+c_n>}.$$

Then, M is a commutative monoid.

Theorem 5.1. *The composition of any CA and the CA with local formula $x^a \in F(N)$ is commutative.*

Proof. Let A be a local formula of CA on the commutative monoid. By Proposition 4.8, $x^a \circ A \equiv A \circ x^a$ holds. Then,

$$\begin{aligned}
T_{x^a}\circ T_A &\equiv T_{x^a\circ A}\\
&\equiv T_{A\circ x^a}\\
&\equiv T_A\circ T_{x^a}
\end{aligned}$$

by Definition 4.1, Proposition 4.2 and Theorem 4.5. □

For CAs over $M = \{x^a | a \in \mathbb{N}^n\}$, and $N = \{x^{a_1}, x^{a_2}, \ldots, x^{a_k}\} \subset M$, there exist n-dimensional CAs with the local formula $x^{a_i} \in F(N)$, whose composition of any CA is commutative. In the case of 2-dimensional CAs over $M = \{x^b | b \in \mathbb{N}^2\}$ and $N = \{x^{<0,0>}, x^{<0,1>}, x^{<1,0>}, x^{<1,1>}\} \subset M$, the composition of any CA and a CA with a local function of Type-1 in Table 3 is commutative. Moreover in the case of elementary CAs over $M = \{x^m | m \in \mathbb{N}\}$ and $N = \{x^0, x^1, x^2\}$, the composition of any CA and the CA with the local function of number 170, 204 or 240 is commutative.

Theorem 5.2. *Let $B, C, D \in N \cup \{\neg x^b | x^b \in N\}$, $x^a \in N$. Each composition of CA with the local formula $\neg x^a \in F(N)$ and CA with the local formula $((B \wedge (C \vee D)) \vee (C \wedge D)) \in F(N)$ is commutative.*

Proof. Let $B, C, D \in N \cup \{\neg x^b | x^b \in N\}$, $x^a \in N$. By Proposition 2.4, Proposition 4.7 and Proposition 4.10,

$$\begin{aligned}\neg x^a \circ ((B \wedge (C \vee D)) \vee (C \wedge D)) &\equiv ((B \vee (C \vee D)) \wedge (C \vee D)) \circ \neg x^a \\ &\equiv ((B \wedge (C \vee D)) \vee (C \wedge D)) \circ \neg x^a.\end{aligned}$$

By Theorem 4.5,

$$\begin{aligned}T_{\neg x^a} \circ T_{((B \wedge (C \vee D)) \vee (C \wedge D))} &\equiv T_{\neg x^a \circ ((B \wedge (C \vee D)) \vee (C \wedge D))} \\ &\equiv T_{((B \wedge (C \vee D)) \vee (C \wedge D)) \circ \neg x^a} \\ &\equiv T_{((B \wedge (C \vee D)) \vee (C \wedge D))} \circ T_{\neg x^a}.\end{aligned}$$

□

For CAs over $M = \{x^a | a \in \mathbb{N}^n\}$, and $N = \{x^{a_1}, x^{a_2}, \ldots, x^{a_n}\} \subset M$, the composition of a CA with a local function of Type-2 and a CA with a local function of Type-3, a local function of Type-4 or a local function of Type-5 in Table 1 is commutative.

Example 5.3. *In the case of 2-dimensional CAs over* $M = \{x^b | b \in \mathbb{N}^2\}$ *and* $N = \{x^{<0,0>}, x^{<0,1>}, x^{<1,0>}, x^{<1,1>}\} \subset M$*, the composition of the CA with the local formula* $LF_A = \neg x^{<1,0>}$ *and the CA with the local formula* $LF_B = (x^{<0,1>} \wedge (x^{<1,0>} \vee x^{<1,1>})) \vee (x^{<1,0>} \wedge x^{<1,1>})$ *is commutative. Figure 1 shows examples of the transitions of their CA and their composition.*

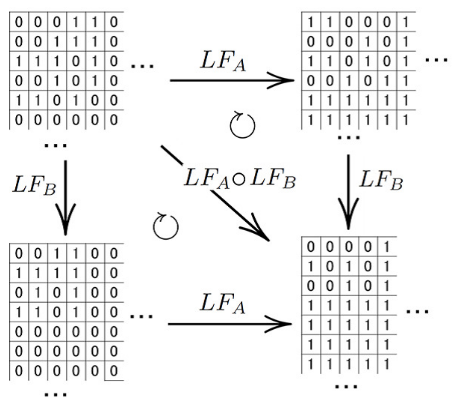

Figure 1: Example of transited configurations

Furthermore, in the case of elementary CAs over $M = \{x^m | m \in \mathbb{N}\}$ and $N = \{x^0, x^1, x^2\}$, the composition of the CA with the local function of 15, 51, or 85 and the CA with the local function of 43, 77, 113, 142, 212, or 232 is commutative.

Theorem 5.4. *Let* $B, C, D \in N \cup \{\neg x^b | x^b \in N\}$, $x^a \in N$. *Each composition of the CA with the local formula* $\neg x^a \in F(N)$ *and the CA with the local formula* $\neg((B \wedge (C \vee D)) \vee (C \wedge D)) \in F(N)$ *is commutative.*

Proof. By Proposition 4.10,

$$\neg x^a \circ \neg((B \wedge (C \vee D)) \vee (C \wedge D)) \equiv \neg((B \wedge (C \vee D)) \vee (C \wedge D)) \circ \neg x^a.$$

By Theorem 4.5,

$$\begin{aligned}T_{\neg x^a} \circ T_{\neg((B \wedge (C \vee D)) \vee (C \wedge D))} &\equiv T_{\neg x^a \circ \neg((B \wedge (C \vee D)) \vee (C \wedge D))} \\ &\equiv T_{\neg((B \wedge (C \vee D)) \vee (C \wedge D)) \circ \neg x^a} \\ &\equiv T_{\neg((B \wedge (C \vee D)) \vee (C \wedge D))} \circ T_{\neg x^a}.\end{aligned}$$

□

For CAs over $M = \{x^a | a \in \mathbb{N}^n\}$, and $N = \{x^{a_1}, x^{a_2}, \ldots, x^{a_n}\} \subset M$, the composition of a CA with a local function of Type-2 and a CA with a local function of Type-6 in Table 1 is commutative. For example, in the case of 2-dimensional CAs over $M = \{x^b | b \in \mathbb{N}^2\}$ and $N = \{x^{<0,0>}, x^{<0,1>}, x^{<1,0>}, x^{<1,1>}\} \subset M$, the composition of the CA with the local formula LF_A and the CA with the local formula $\neg LF_B$ is commutative for the local formulas in Example 5.3. Additionally, in the case of elementary CAs over $M = \{x^m | m \in \mathbb{N}\}$ and $N = \{x^0, x^1, x^2\}$, the composition of the CA with the local function of 15, 51, or 85 and the CA with the local function of 23 or 43 is commutative.

Theorem 5.5. *Let $B, C, D \in N \cup \{\neg x^b | b \in \mathbb{N}^n\}$, $x^a \in N$. Each composition of the CA with the local formula $\neg x^a \in F(N)$ and the CA with either local formula $(B + C + D) \in F(N)$ or $\neg(B + C + D) \in F(N)$is commutative.*

Proof. By Proposition 4.10,

$$\neg x^a \circ \neg(B + C + D) \equiv \neg(B + C + D) \circ \neg x^a.$$

By Theorem 4.5,

$$\begin{aligned} T_{\neg x^a} \circ T_{\neg(B+C+D)} &\equiv T_{\neg x^a \circ \neg(B+C+D)} \\ &\equiv T_{\neg(B+C+D) \circ \neg x^a} \\ &\equiv T_{\neg(B+C+D)} \circ T_{\neg x^a}. \end{aligned}$$

By Proposition 4.10,

$$\neg x^a \circ (B + C + D) \equiv (B + C + D) \circ \neg x^a.$$

By Theorem 4.5,

$$\begin{aligned} T_{\neg x^a} \circ T_{(B+C+D)} &\equiv T_{\neg x^a \circ (B+C+D)} \\ &\equiv T_{(B+C+D) \circ \neg x^a} \\ &\equiv T_{(B+C+D)} \circ T_{\neg x^a}. \end{aligned}$$

□

For CAs over $M = \{x^a | a \in \mathbb{N}^n\}$, and $N = \{x^{a_1}, x^{a_2}, \ldots, x^{a_n}\} \subset M$, the composition of a CA with a local function of Type-2 and a CA with a local function of Type-7 or a local function of Type-8 in Table 1 is commutative. For example, in the case of 3-dimensional CAs over $M = \{x^b | b \in \mathbb{N}^3\}$ and $N = \{x^{<0,0,0>}, x^{<0,0,1>}, x^{<0,1,0>}, x^{<0,1,1>}, x^{<1,0,0>}, x^{<1,0,1>}, x^{<1,1,0>}, x^{<1,1,1>}\} \subset M$, the composition of the CA with the local formula $\neg x^{<1,0,1>}$ and the CA with the local formula $x^{<0,1,0>} + x^{<1,0,0>} + x^{<1,1,1>}$ is commutative and it is the composition of the CA with the local formula $\neg x^{<0,0,1>}$ and the CA with the local formula $x^{<0,0,1>} + x^{<0,1,1>} + x^{<1,1,0>}$. Furthermore, in the case of elementary CAs over $M = \{x^m | n \in \mathbb{N}\}$ and $N = \{x^0, x^1, x^2\}$, the composition of the CA with the local function of 15, 51, or 85 and the CA with the local function of 77, 133, 142, 212, or 232 is commutative.

Assuming M is a monoid not confined to a commutative monoid, the following proposition holds.

Proposition 5.6. *Let CA_A and CA_B with local formulae A and B on monoid M be reversible. If the multiplication of A and B is commutative, then the multiplication of B^{-1} and A^{-1} is commutative.*

Proof.

$$\begin{aligned} (A \circ B) \circ (B^{-1} \circ A^{-1}) &\equiv e \\ (A^{-1} \circ B^{-1}) \circ (A \circ B) &\equiv (A^{-1} \circ B^{-1}) \circ (B \circ A) \\ &\equiv e \end{aligned}$$

By Proposition 4.7, $(A^{-1} \circ B^{-1}) \equiv (B^{-1} \circ A^{-1})$. □

6 Conclusion

In this paper, we examined the logical formulae on a commutative monoid as the local functions of n-dimensional CAs. We discussed the commutativity of the multiplication of formulae and described the conditions for formulae that satisfy the commutativity of the composition of CAs of local functions using one cell in its neighborhood on commutative monoids. We also provided the examples of the compositions that satisfy the commutativity for n-dimensional CAs on commutative monoids.

We focused on $M = \{x^a | a \in \mathbb{N}^n\}$ in this paper. We guess that some of those properties hold for CAs on other monoids. One of the future works is to check them. Furthermore questions concerning CAs on monoids remain to be answered in future research. First, we need to formulate more computational laws of formulae on monoids. Second, we should clarify conditions for CAs that cannot be created by compositions of CAs. Finally, we need to study which of reversible CAs on monoids exist. It is also necessary to determine whether well-known theorems and properties of CAs on groups apply for CAs over monoids.

References

[1] H. Kleine Büning and T. Lettmann. (1999). Propositional Logic: Deduction and Algorithms, Cambridge Tracts in Theoretical Computer Science, Cambridge.

[2] T. Ceccherini-Silberstein and M. Coornaert. (2010). Cellular Automata and Groups, Springer Monographs in Mathematics. Springer-Verlag, Berlin.

[3] G. A. Hedlund, (1969). Endomorphisms and automorphisms of the shift dynamical system, Mathematical Systems Theory 3 (4), pages 320-375.

[4] S. Inokuchi, T. Ito, M. Fujio and Y. Mizoguchi. (2014). A formulation of composition for cellular automata on groups, IEICE Transactions on Information and Systems E 97-D (3), pages 448-454.

[5] S. Inokuchi, T. Ishida and Y. Kawahara. (2014). Propositional logic and cellular automata, RIMS Kokyuroku Volume 1894, Research Institute for Mathematical Sciences, Kyoto University, pages 19-27.

[6] T. Ishida, S. Inokuchi and Y. Kawahara. (2016) Propositional Logic and Cellular Automata on Monoids, Journal of Cellular Automata, 12.1-2, pages 27 - 45.

[7] C. Moore and A. A. Drisko. (1996). Algebraic properties of the block transformation on cellular automata, Complex Systems, 10(3), pages 185-194.

[8] J. von Neumann. (1966). The Theory of Self-Reproducing Automata (A. Burks ed.), University of Illinois Press.