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Solving Distance-constrained Labeling Problems for Small Diameter Graphs via TSP*

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Abstract

For an undirected graph $G = (V, E)$ and a k-non-negative integer vector $\boldsymbol{p} = (p_1, \ldots, p_k)$, a mapping $l \colon V \to \mathbb{N} \cup \{0\}$ is called an $L(\boldsymbol{p})$-labeling of G if $|l(u) - l(v)| \geq p_d$ for any two distinct vertices $u, v \in V$ with distance d, and the maximum value of $\{l(v) \mid v \in V\}$ is called the span of l. Originally, $L(\boldsymbol{p})$-labeling of G for $\boldsymbol{p} = (2, 1)$ is introduced in the context of frequency assignment in radio networks, where 'close' transmitters must receive different frequencies and 'very close' transmitters must receive frequencies that are at least two frequencies apart so that they can avoid interference. $L(\boldsymbol{p})$-Labeling is the problem of finding the minimum span $\lambda_{\boldsymbol{p}}$ among $L(\boldsymbol{p})$-labelings of G, which is NP-hard for every non-zero $\boldsymbol{p}$. $L(\boldsymbol{p})$-Labeling is well studied for specific $\boldsymbol{p}$'s; in particular, many (exact or approximation) algorithms for general graphs or restricted classes of graphs are proposed for $\boldsymbol{p} = (2, 1)$ or more generally $\boldsymbol{p} = (p, q)$. Unfortunately, most algorithms strongly depend on the values of $\boldsymbol{p}$, and it is not apparent to extend algorithms for $\boldsymbol{p}$ to ones for another $\boldsymbol{p}'$ in general. In this paper, we give a simple polynomial-time reduction of $L(\boldsymbol{p})$-Labeling on graphs with a small diameter to Metric (Path) TSP, which enables us to use numerous results on (Metric) TSP. On the practical side, we can utilize various high-performance heuristics for TSP, such as Concordo and LKH, to solve our problem. On the theoretical side, we can see that the problem for any $\boldsymbol{p}$ under this framework is 1.5-approximable, and it can be solved by the Held-Karp algorithm in $O(2^n n^2)$ time, where n is the number of vertices, and so on.

Keywords: Frequency Assignment, Distance-constrained Labeling, $L(p_1, \ldots, p_k)$-Labeling, TSP, Graph Diameter, Parameterized Complexity

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1 Introduction

For an undirected graph G with n vertices and m edges, and a k-nonnegative integer vector $\boldsymbol{p} = (p_1, \ldots, p_k)$, a mapping $l\colon V \to \mathbb{N} \cup \{0\}$ is an $L(\boldsymbol{p})$-labeling of G if $|l(u) - l(v)| \geq p_d$ for any two distinct vertices $u, v \in V$ with distance d, the maximum value of $\{l(v) \mid v \in V\}$ is called the span of l. The notion of $L(\boldsymbol{p})$-labeling for $\boldsymbol{p} = (2, 1)$ can be seen in Hale [18] and Roberts [30] in the context of frequency assignment in radio networks, where 'close' transmitters must receive different frequencies and 'very close' transmitters must receive frequencies that are at least two frequencies apart so that they can avoid interference. $L(\boldsymbol{p})$-Labeling is the problem of finding the minimum span $\lambda_{\boldsymbol{p}}$ among $L(\boldsymbol{p})$-labelings of G, which is NP-hard for every non-zero $\boldsymbol{p}$. Since $L(\boldsymbol{p})$-Labeling for $k = 1$ is the ordinary coloring problem, the cases of $k \geq 2$ are essential to study $L(\boldsymbol{p})$-Labeling under its name. In particular, the problem for $\boldsymbol{p} = (p_1, p_2) = (p, q)$ is called the $L(p, q)$-Labeling problem and intensively and extensively studied.

Among infinite settings of (p, q), probably $L(2, 1)$-Labeling is most studied. It is shown that $L(2, 1)$-Labeling is NP-hard even for restricted classes of graphs, such as planar graphs, bipartite graphs, chordal graphs [6], graphs with diameter 2 [17], and graphs of tree-width 2 [13]. In contrast, only a few graph classes are known to be solvable in polynomial time. For example, $L(2, 1)$-Labeling can be solved in polynomial time for paths, cycles, wheels [17], co-graphs, and trees [8, 21]. These algorithms are straightforward (paths, cycles, wheels) or strongly depend on the properties of graphs (co-graphs and trees). In fact, the NP-hardness for graphs of tree-width 2 implies that the polynomial-time solvability for trees (graphs of tree-width 1) depends on not a tree-like structure but the tree structure itself; it might be difficult to extend or generalize algorithms for trees to superclasses of trees. Note that the algorithm of [21] for trees is quite involved though its running time is linear. Furthermore, $L(p, q)$-Labeling is NP-hard even for trees, if p and q do not have a common divisor.

Another direction of research for intractable problems is to design exact exponential-time algorithms whose bases or exponents are small. For example, Junosza-Szaniawski et al. [25] present an algorithm for $L(2, 1)$-Labeling whose running time is $O(2.6488^n)$, which is currently the fastest. This algorithm uses the exponential size of memories. The current fastest exact algorithm with polynomial space for $L(2, 1)$-Labeling is proposed by Junosza-Szaniawski et al. [26], and it runs in $O(7.4922^n)$ time. These algorithms are specialized in $L(2, 1)$-Labeling. As more generalized algorithms, Cygan and Kowalik presented an exact algorithm for a more general labeling problem, called *channel assignment problem*. It is based on the fast zeta transform in combination with the inclusion-exclusion principle [11]. The algorithm solves $L(p, q)$-Labeling in $O^*((\max\{p, q\} + 1)^n)$ time and $L(2, 1)$-Labeling in $O^*(3^n)$ time, where polynomial factors are omitted in O^* notation.

In summary, $L(\boldsymbol{p})$-Labeling is well studied in the fields of algorithm design, but most of the developed algorithms are tailored to $\boldsymbol{p}$ and graph classes, and it is hard to generalize them.

1.1 Our contribution

In this paper, we address the $L(\boldsymbol{p})$-Labeling problem on graphs with a small diameter, which is known to be NP-hard. Our approach is simple; we just solve the problem via TSP. Namely, our main contribution is an $O(nm)$-time reduction from $L(\boldsymbol{p})$-Labeling for graph G with diameter at most the dimension of $\boldsymbol{p}$, say k, to Metric Path Traveling Salesman Problem (TSP) under the assumption that $p_{\max} \leq 2p_{\min}$, where $p_{\min} = \min\{p_1, \ldots, p_k\}$ and $p_{\max} = \max\{p_1, \ldots, p_k\}$. Note that the most well-studied setting $\boldsymbol{p} = (2, 1)$ satisfies this condition. Although this reduction is available only for graphs with a small diameter and $\boldsymbol{p}$ satisfying the above condition, it enables us to use numerous results of (Metric) TSP.

On the practical side, since many practical algorithms for (Metric) TSP have been developed, they can be applied to solve $L(\boldsymbol{p})$-Labeling for graphs with a small diameter with a minor modification. For example, the Lin-Kernighan heuristic for symmetric TSP [29] and its variants are known to have outstanding performance, and there are several excellent implementations [1, 24]. Such implementations can be used to solve our problems as engines practically.

On the theoretical side, the reduction leads to several algorithms with performance guarantees,

such as an $O(2^n n^2)$-time algorithm and a 1.5-approximation algorithm for $L(\boldsymbol{p})$-Labeling if the diameter of an input graph is at most k and if $p_{\max} \leq 2p_{\min}$. Both of the results imply that a small diameter and the setting $\boldsymbol{p}$ may make the problem easier; it is only known that $L(p,q)$-Labeling for general graphs can be solved in $O^*((\max\{p,q\}+1)^n)$ time and be $O(\min\{\Delta, \sqrt{n}+p/q\})$-approximable in polynomial time, where Δ is maximum degree. Particularly, in case of $k=2$, our reduction reduces the problem (i.e, $L(p,q)$-Labeling) to Path TSP with 2-valued edge weights, which can be solved via Partition into Paths. Since Partition into Path is known to be fixed-parameter tractable for modular-width [16], so is our problem. On the other hand, we point out that $L(p,q)$-Labeling for graphs with diameter 2 is W[1]-hard for clique-width, which could show a frontier between fixed parameter (in)tractability.

In passing, we can show that $L(1,\ldots,1)$-Labeling on general graphs is fixed-parameter tractable for modular-width. Although the parameterized complexity of $L(p,q)$-Labeling for modular-width remains open in general, $L(\boldsymbol{p})$-Labeling becomes $p_{\max}$-approximable in FPT time for modular-width by the FPT result for $L(1,\ldots,1)$-Labeling.

Finally, we investigate the polynomial-time solvability of $L(2,1)$-Labeling on subclasses of split graphs whose diameters are at most 3. Though the problem is NP-complete on split graphs of diameter 2 [6], we show that it can be solved in linear time on block split graphs and threshold graphs.

1.2 Related work

1.2.1 Distance-Constrained labeling

The original notion of distance-constrained labeling can be seen in Hale [18] and Roberts [30] in the context of frequency assignment. In frequency assignment, 'close' transmitters must receive different frequencies, and 'very close' transmitters must receive frequencies that are at least two frequencies apart to avoid interference. Then, Griggs and Yeh formally introduced the notion of $L(p,q)$-labeling in [17]. Since p and q could be any natural numbers, there are infinite settings of $L(p,q)$-labeling, but $L(2,1)$-labeling is most studied. One of the reasons is the context of more general frequency assignment because the setting explained above is interpreted as $L(2,1)$-labeling. In the context of frequency assignment, it is natural to consider the setting of $p \geq q$. Also, $q=1$ might be natural because it decides the unit. Another reason why $L(2,1)$ is most popular is that the setting of $p=2$ and $q=1$ seems the most natural and fundamental among the settings represented by $L(p,q)$-labeling. Indeed, $L(1,1)$-labeling of G is equivalent to the ordinary coloring on the square of G; we do not need to study $L(1,1)$-labeling itself in this name.

As introduced in the previous sections, the $L(p,q)$-labeling problem or specifically the $L(2,1)$-labeling problem is NP-hard even for restricted classes of graphs. Thus polynomial-time algorithms for particular classes of graphs and exact exponential-time algorithms are developed. We list here other results than those mentioned before. As for approximation, $L(p,q)$-Labeling is NP-hard to approximate within factor better than $n^{\frac{1}{2}-\varepsilon}$. On the other hand, there is an asymptotically tight $O(\min\{\Delta, \sqrt{n}+p/q\})$-approximation algorithm where Δ is the maximum degree of G [19].

For the parameterized complexity, the $L(2,1)$-Labeling problem is fixed-parameter tractable for vertex cover number [14], clique-width plus maximum degree, or twin cover number plus maximum clique size [20]. Although it is less critical to study $L(1,\ldots,1)$-Labeling (we write $L(\mathbf{1})$-Labeling hereafter) in this name, $L(\mathbf{1})$-Labeling can be used for approximating $L(\boldsymbol{p})$-Labeling; $L(\mathbf{1})$-Labeling yields $p_{\max}$-approximation of $L(\boldsymbol{p})$-Labeling, $p_{\max} = \max_{d\in[k]} p_d$. For this reason, we are interested in the complexity of $L(\mathbf{1})$-Labeling or Coloring of powers of graphs. It is known that $L(1,1)$-Labeling is W[1]-hard for the tree-width [14], even though the ordinary Coloring is FPT, but $L(\mathbf{1})$-Labeling is in XP for clique-width [31], which implies that it is in XP for tree-width. Hanaka et al. also show that $L(1,1)$-Labeling is fixed-parameter tractable when parameterized by twin cover number [20].

The generalized setting, $L(\boldsymbol{p})$, is also studied but is less popular. Bertossi and Pinotti present approximation algorithms of $L(\boldsymbol{p})$-Labeling for trees and interval graphs [5]. $L(\boldsymbol{p})$-Labeling is fixed-parameter tractable for the neighborhood diversity, $p_{\max}$, plus k [12]. Further related work for

$L(\boldsymbol{p})$-Labeling can be found in the following surveys [7, 22].

1.2.2 (Metric Path) TSP

Traveling Salesman Problem (TSP) might be the most studied combinatorial optimization problem from both practical and theoretical points of view. Thus, we here list only a few of the results.

On the practical side, an enormous number of works have been devoted to developing efficient algorithms for TSP for a long time. For example, as mentioned before, implementations of the Lin-Kernighan type algorithms [29] have outstanding performance, and it was reported even in 2003 [3] that an implementation of the chained Lin-Kernighan can constantly find near-optimal solutions for instances with 100,000 cities or more. Moreover, some implementations, such as Concorde and LKH [1, 2], are available on the Web. Developments are continuing, and improvements are still reported [33, 37].

On the theoretical side, TSP has been studied from various aspects. For example, the Held-Karp algorithm with time complexity $O^*(2^n)$ was proposed in 1962 [4, 23], and the existence of an exact algorithm with time complexity $O^*(c^n)$ for some $c < 2$ is still open [35]. For approximation, the general symmetric TSP has no approximation algorithm unless P=NP, whereas the Metric TSP, which is a restricted version of TSP whose edge-weights satisfy the triangle inequality, is known to be 1.5-approximable by the Christofides algorithm [9]. Recently, this bound has been slightly improved by a randomized algorithm whose approximation ratio is at most $1.5 - 10^{-36}$ [27]. Note that our reduction is not to Metric TSP but to Metric Path TSP. Naive applications of algorithms for Metric TSP to Metric Path TSP do not preserve approximation guarantees, though it is shown that α-approximation algorithm for TSP can be used to obtain an $(\alpha + \varepsilon)$-approximation solution of Path TSP for arbitrary $\varepsilon > 0$ [34]. For Metric Path TSP, Zenklusen recently gives a deterministic 1.5-approximation algorithm [36]. By combining the results on [27] and [34], a randomized algorithm can obtain an approximate solution whose ratio is slightly better than 1.5.

2 Preliminaries

2.1 Definitions and notations

We assume basic knowledge of graph theory. Throughout the paper, let $G = (V, E)$ be an undirected and connected graph where $n = |V|$ and $m = |E|$. The *distance* between two vertices u, v in G is defined by the shortest path length between u and v in G and denoted by $\mathrm{dist}_G(u, v)$. The *diameter* of G is defined by $\mathrm{diam}(G) = \max_{u,v \in V} \mathrm{dist}_G(u, v)$. For a vertex $v \in V$, let $N_G(v) = \{u \in V \mid \{u, v\} \in E\}$ denote the set of adjacent vertices of v in G. The *degree* of a vertex v in G is defined by $|N_G(v)|$. For a vertex subset $S \subseteq V$, $G[S]$ is defined as the subgraph induced by S, that is, $G[S] = (S, E[S])$, where $E[S] = \{\{u, v\} \in E \mid \{u, v\} \subseteq S\}$. The complement graph of G is denoted by $\overline{G}$, i.e., $\overline{G} = (V, \bar{E})$, where $\bar{E} = \binom{V}{2} \setminus E$. Also, the k-th power of graph G, denoted by G^k, is defined by the vertex set V and the edge set E^k, where $\{u, v\} \in E^k$ if and only if $\mathrm{dist}_G(u, v) \leq k$. Given a positive integer k, we define $[k] = \{1, 2, \ldots, k\}$. Also, given nonnegative integer k and $k' (\geq k)$, we define $[k, k'] = \{k, k + 1, \ldots, k'\}$. For an integer vector $\boldsymbol{p} = (p_1, \ldots, p_k)$, we define $p_{\min} = \min\{p_1, \ldots, p_k\}$ and $p_{\max} = \max\{p_1, \ldots, p_k\}$. Let $\mathbf{1} = (1, \ldots, 1)$ be a vector such that each element is 1. A subset S of the vertex set V is called a *clique* of G if every two vertices in S are adjacent in G. A subset S of the vertex set V is called an *independent set* of G if every two vertices in S are not adjacent in G.

2.2 Graph parameters

A vertex subset $M \subseteq V$ is a *module* of a graph G if any pair of u, v in M satisfies that $N_G(u) \setminus M = N_G(v) \setminus M$.

Definition 1 (Modular-width)**.** *A graph $G = (V, E)$ has* modular-width *at most ℓ (≥ 2) if it satisfies (i) $|V| \leq \ell$, or (ii) there is a partition $(V_1, \ldots, V_\ell)$ of V such that for each $i \in [\ell]$, V_i is a*

module and $G[V_i]$ has modular-width *at most ℓ. The minimum ℓ such that G has modular-width at most ℓ is denoted by* $\mathtt{mw}(G)$.

There is a polynomial-time algorithm that computes $\mathtt{mw}(G)$ and its decomposition [32].

Definition 2 (Neighborhood diversity)**.** *A graph $G = (V, E)$ has* neighborhood diversity *at most ℓ if there is a partition $(V_1, \ldots, V_\ell)$ of V such that every pair of vertices u, v in V_i satisfies $N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\}$ for each $i \in [\ell]$. The minimum ℓ is denoted by* $\mathtt{nd}(G)$.

Note that each of V_i's in Definition 2 is a module of G and it forms either an independent set or a clique. As with modular-width, there is a polynomial-time algorithm for computing $\mathtt{nd}(G)$ and its partition [28].

Proposition 1. *For any graph $G = (V, E)$, $\mathtt{mw}(G) = \mathtt{mw}(\overline{G})$ holds.*

Proof. It is sufficient to show that if G has modular-width at most ℓ, then $\overline{G}$ has modular-width at most ℓ. We show this claim by induction on the number of vertices n. First, if $n \leq \ell$, then both G and $\overline{G}$ clearly satisfy condition (i), so the claim holds.

Next, assume that $n > \ell$ and that the claim holds for any graph whose number of vertices is less than n. Let $(V_1, \ldots, V_t)$ be a partition of V such that each V_i is a module and $G[V_i]$ has modular-width at most ℓ. Note that $t \leq \ell$. Then, for each pair of $u, v \in V_i$, it holds that:

$$\begin{aligned} N_{\overline{G}}(u) \setminus V_i &= (V \setminus N_G(u)) \setminus V_i \\ &= (V \setminus N_G(v)) \setminus V_i \\ &= N_{\overline{G}}(v) \setminus V_i. \end{aligned}$$

Therefore, V_i is module of $\overline{G}$. Furthermore, since $\overline{G}[V_i] = \overline{G[V_i]}$, $\overline{G}[V_i]$ has modular-width at most ℓ by the assumption of induction. Therefore, $\overline{G}$ satisfies condition (ii) of Def.1. □

Proposition 2. *For any connected graph $G = (V, E)$, $\mathtt{nd}(G^2) \leq \mathtt{mw}(G)$ holds, where G^2 is the second power of G.*

Proof. If $|V| \leq \mathtt{mw}(G)$, we are done as $\mathtt{nd}(G^2) \leq |V|$. Otherwise, consider a partition $(V_1, \ldots, V_\ell)$ of V such that V_i is a module for each $i \in [\ell]$ where $\ell \leq \mathtt{mw}(G)$. Since G is connected, any module is adjacent to at least one module, and vertices between two modules are completely joined; that is, for the two modules V_i and V_j, there is an edge $\{u, v\}$ between any pair of $u \in V_i$ and $v \in V_j$. Thus, the distance of each pair of vertices in a module is at most 2, and hence each module forms a clique in G^2. Furthermore, for each pair of $u, v \in V_i$, $N_{G^2}(u) \setminus V_i = N_{G^2}(v) \setminus V_i$ follows from $N_G(u) \setminus V_i = N_G(v) \setminus V_i$. Therefore, $N_{G^2}(u) \setminus \{v\} = N_{G^2}(v) \setminus \{u\}$ holds, which implies $\mathtt{nd}(G^2) \leq \mathtt{mw}(G)$. □

Finally, we introduce the clique-width $\mathtt{cw}(G)$ of G, which is a more general graph parameter than tree-width, modular-width, and neighborhood diversity. Namely, if some problem is not in FPT for tree-width, modular-width or neighborhood diversity, it is also not in FPT for clique-width. It is defined by some tree structures like tree-width.

Definition 3 (Clique-width)**.** *Let w be a positive integer. The w-graph is a vertex-labeled graph with $\{1, 2, \ldots, w\}$. The clique-width $\mathtt{cw}(G)$ of a graph G is the minimum integer w such that G can be constructed by applying the following operations repeatedly:*

- *Add a new vertex labeled by $i \in \{1, \ldots, w\}$.*
- *Take a disjoint union of two w-graphs G_1 and G_2.*
- *Take two labels i and j and then add edges between every pair of vertices with label i and vertices with label j.*
- *Relabel the vertices labeled by i label j.*

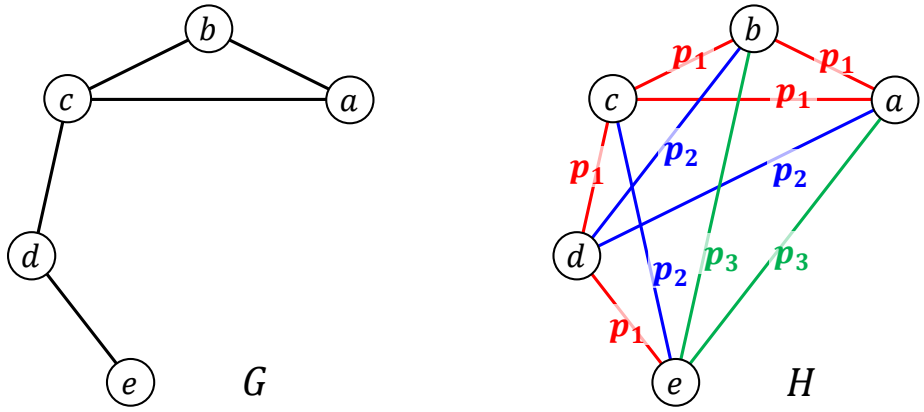

Figure 1: The construction of H for $L(p_1, p_2, p_3)$-Labeling on G with diameter 3.

Clique-width is a well-studied graph parameter, and many results are known. For example, cographs are the graph class of clique-width at most 2. We refer readers to [10]. In order to show the W[1]-hardness of $L(2,1)$-Labeling on graphs with diameter 2 when parameterized by clique-width in Section 4, we prove that Hamiltonian Path is W[1]-hard.

Theorem 1. Hamiltonian Path *is W[1]-hard for clique-width.*

Proof. We reduce Hamiltonian Cycle, which is W[1]-hard for clique-width [15]. Given a graph $G = (V, E)$ of clique-width $\mathtt{cw}(G)$, pick arbitrary vertex v and add a new vertex v' that is adjacent to vertices in $N(v)$. That is, v and v' are false twins. Then we further add two vertices w, w' that are adjacent to v and v', respectively. It is easily seen that G has a hamiltonian cycle if and only if the constructed graph G' has a hamiltonian path from w to w'. Since adding a vertex that is a false twin for some vertex to G does not change the clique-width and adding a leaf vertex increases the clique-width by at most 2, $\mathtt{cw}(G') \leq \mathtt{cw}(G) + 4$ holds. This completes the proof. □

3 Main results

In this section, we show a polynomial-time reduction from $L(\boldsymbol{p})$-Labeling to Metric Path TSP. Path TSP is the problem to finding a hamiltonian path of minimum weight on an edge-weighted complete graph. Furthermore, Metric Path TSP is the restricted version of Path TSP such that the edge-weights of the input graph satisfy the triangular inequality.

Theorem 2. *If* $p_{\max} \leq 2p_{\min}$, $L(\boldsymbol{p})$-Labeling *on graphs of diameter at most* k *can be reduced to* Metric Path TSP *in* $O(nm)$ *time.*

Proof. First, we define an edge-weighted complete graph $H = (V, \binom{V}{2}, w)$ from an input graph G (see Figure 1). For a pair of vertices $u, v \in V$ with $\mathrm{dist}_G(u, v) = d$, the edge weight of $\{u, v\}$ in H is defined by $w(u, v) = p_d$. Note that since $\mathrm{diam}(G) \leq k$, $w(u, v)$ is well-defined. Furthermore, $p_{\min} \leq w(u, v) \leq 2p_{\min}$ holds by $p_d \leq 2p_{\min}$ for each $d \in [k]$, and thus w satisfies the triangle inequality.

For a permutation $\pi \colon V \to [n]$, we say that an $L(\boldsymbol{p})$-labeling ℓ is an $L(\boldsymbol{p})$-*labeling for* π if it satisfies $\ell\left(\pi^{-1}(1)\right) \leq \ell\left(\pi^{-1}(2)\right) \leq \cdots \leq \ell\left(\pi^{-1}(n)\right)$. We denote by $\lambda_{\boldsymbol{p}}(G, \pi)$ the minimum span among all of $L(\boldsymbol{p})$-labelings for π. Here, we observe that any minimum $L(\boldsymbol{p})$-labelings for π satisfies $\ell\left(\pi^{-1}(1)\right) = 0$. If not, we obtain another labeling ℓ' such that $\ell'((\pi^{-1}(i)) = \ell((\pi^{-1}(i)) - 1$, which contradicts the minimality of ℓ.

Given a permutation π, let ℓ be an $L(\boldsymbol{p})$-labeling for π with minimum span $\lambda_{\boldsymbol{p}}(G, \pi)$ on G. In the following, we denote $v_i = \pi^{-1}(i)$ and $w_{i,j} = w(v_i, v_j)$ for simplicity. Then we show the following key claim, which implies that $\ell(v_i)$ is the length (sum of weights) of path $(v_1, v_2, \cdots, v_i)$ on H.

Claim 1. *For the edge-weighted complete graph H, the labeling ℓ satisfies that $\ell(v_i) = \sum_{t=1}^{i-1} w_{t,t+1}$ for any $i \in [n]$.*

Proof. We prove the claim by induction on i. As the base case, we have that $\ell(v_1) = 0$. Furthermore, we consider the case of $i = 2$. Since ℓ is a minimum $L(\boldsymbol{p})$-labeling for π, it satisfies that $0 = \ell(v_1) \leq \ell(v_2) \leq \cdots \leq \ell(v_n)$. Since $0 = \ell(v_1) \leq \ell(v_2)$, we have $\ell(v_2) \geq \ell(v_1) + p_{\mathrm{dist}_G(v_1,v_2)} = w_{1,2}$. Moreover, $\ell(v_2) \leq w_{1,2}$ follows from $\ell(v_2) \leq \cdots \leq \ell(v_n)$ and the minimality of ℓ. Thus, the claim holds when $i = 2$.

In the induction step, assume that the claim holds for each $j \in [i-1]$. By the minimality of ℓ and $\ell(v_1) \leq \cdots \leq \ell(v_n)$, the label of v_i can be expressed as:

$$\begin{aligned}\ell(v_i) &= \min\left\{x \mid x \geq \ell(v_j) + p_{\mathrm{dist}_G(v_j,v_i)}, \forall j \in [i-1]\right\} \\ &= \min\left\{x \mid x \geq \ell(v_j) + w_{j,i}, \forall j \in [i-1]\right\} \\ &= \max_{j\in[i-1]} \left\{\ell(v_j) + w_{j,i}\right\}.\end{aligned}$$

For each $j \in [i-2]$, it holds that

$$\begin{aligned}\ell(v_{i-1}) - \ell(v_j) &= \sum_{t=1}^{i-2} w_{t,t+1} - \sum_{t=1}^{j-1} w_{t,t+1} \\ &= \sum_{t=j}^{i-2} w_{t,t+1} \\ &\geq w_{i-2,i-1} \geq p_{\min}.\end{aligned}$$

Furthermore, $w_{i-1,i} - w_{j,i} \geq p_{\min} - 2p_{\min} = -p_{\min}$ holds. Thus, for any $j \in [i-2]$, we have:

$$\begin{aligned}&(\ell(v_{i-1}) + w_{i-1,i}) - (\ell(v_j) + w_{j,i}) \\ &= (\ell(v_{i-1}) - \ell(v_j)) + (w_{i-1,i} - w_{j,i}) \\ &\geq p_{\min} - p_{\min} = 0.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\ell(v_i) &= \max_{j\in[i-1]} \left\{\ell(v_j) + w_{j,i+1}\right\} \\ &= \ell(v_{i-1}) + w_{i-1,i} \\ &= \sum_{t=1}^{i-1} w_{t,t+1}.\end{aligned}$$

□

Claim 1 means that $\lambda_{\boldsymbol{p}}(G,\pi) = l(v_n)$ is equivalent to the length of the hamiltonian path π on H. Since $\lambda_{\boldsymbol{p}}(G) = \min_\pi \{\lambda_{\boldsymbol{p}}(G,\pi)\}$, Path TSP on H is equivalent to $L(\boldsymbol{p})$-Labeling on G.

Finally, we discuss the running time of the reduction. For the construction of H, we create the distance matrix of G. This can be done in $O(nm)$ time by the breadth-first search for each vertex. We then construct the weighted adjacency matrix of H from the distance matrix of G. Clearly, it can be constructed in $O(n^2)$ time. Thus, the total running time is $O(nm) + O(n^2) = O(nm)$. □

As a corollary, we can obtain an optimal solution in $O(2^n n^2)$ time and a 1.5-approximate solution in polynomial time by applying algorithms for Metric Path TSP proposed in [23] and [36], respectively, after the above reduction.

Corollary 1. *If $p_{\max} \leq 2p_{\min}$, $L(\boldsymbol{p})$-Labeling on graphs of diameter at most k can be solved in $O(2^n n^2)$ time. Furthermore, it is approximable within 1.5 in polynomial time.*

Further observation shows that our problem is fixed-parameter tractable for modular-width.

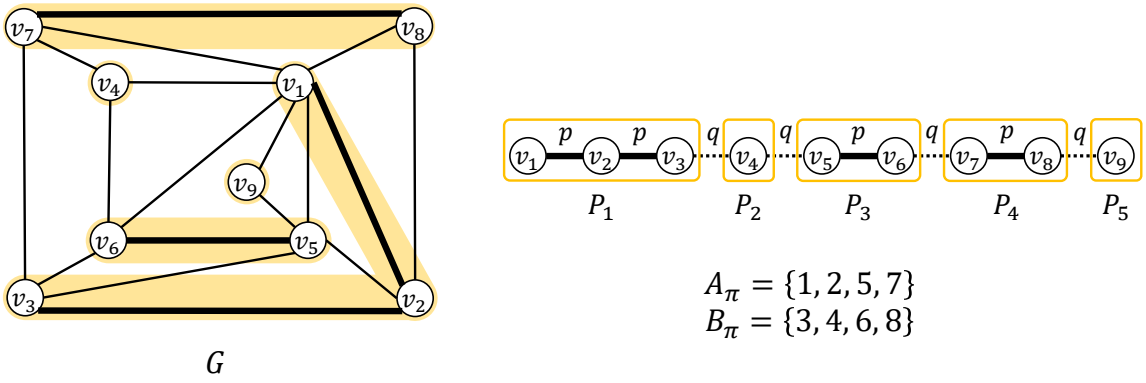

Figure 2: Paths $P_1, \ldots, P_5$ consisting of only edges of weight p along π correspond to paths in G.

Corollary 2. *The $L(p,q)$-Labeling problem on graphs of diameter at most 2 is fixed-parameter tractable for modular-width.*

Proof. Let G be a graph of diameter at most 2 and H be the weighted complete graph obtained from G as in Theorem 2. Notice that the weight of an edge in H is either p or q.

First, we consider the case that $p \leq q$. For a permutation π of V, we define:

$$A_\pi = \{i \in [n-1] \mid w_{i,i+1} = p\}$$
$$B_\pi = \{i \in [n-1] \mid w_{i,i+1} = q\}.$$

Note that $\{\pi^{-1}(i), \pi^{-1}(i+1)\}$ for $i \in A_\pi$ corresponds to an edge in E.

Since the weight of an edge in H is either p or q, the following equation holds:

$$\begin{aligned}\lambda_{\boldsymbol{p}}(G,\pi) &= \sum_{i=1}^{n-1} w_{i,i+1} = \sum_{i \in A_\pi} p + \sum_{i \in B_\pi} q \\ &= (n-1)p + (q-p)\,|B_\pi|\,.\end{aligned}$$

Therefore, we have $\lambda_{\boldsymbol{p}}(G) = (n-1)p + (q-p)\min_\pi |B_\pi|$. Since n, p, q are constant, solving $L(\boldsymbol{p})$-Labeling for G is equivalent to finding π that minimizes $|B_\pi|$ on H.

Here, let $P_1, \ldots, P_s$ be paths along π such that each P_i contains only edges with weight p (see Figure 2). Note that some P_i could be one vertex. By the definition of such paths, $s = |B_\pi| + 1$. We observe that edges in P_i corresponds to edges in G. Thus, minimizing $|B_\pi|$ on H is equivalent to the Partition into Paths problem, which is the problem to minimize the number of paths that partition V in G. This can be computed in $f(\mathtt{mw}(G))n^{O(1)}$ time [16].

For the case that $p > q$, we can similarly solve $L(p,q)$-Labeling by computing Partition into Paths on the complementary graph $\overline{G}$ of G. Since $\mathtt{mw}(\overline{G}) = \mathtt{mw}(G)$ by Proposition 1, it can also be computed in $f(\mathtt{mw}(G))n^{O(1)}$ time. □

4 Related results

4.1 Parameterized complexity

In the previous section, we showed that $L(p,q)$-Labeling is fixed-parameter tractable for modular-width on graphs of diameter 2. In this section, we first point out that $L(2,1)$-Labeling is W[1]-hard for clique-width even on graphs of diameter 2.

Theorem 3. *$L(2,1)$-Labeling on graphs with diameter 2 is W[1]-hard for clique-width.*

Proof. In [17], Griggs and Yeh give a reduction from Hamiltonian Path to $L(2,1)$-Labeling on graphs with diameter 2. Given a graph $G = (V, E)$ as an instance of Hamiltonian Path, the reduced graph of $L(2,1)$-Labeling is constructed by taking the complementary graph $\overline{G}$ of G and adding a universal vertex x that is adjacent to all the vertices in V. Since $\mathtt{cw}(\overline{G}) \leq 2\mathtt{cw}(G)$ holds for any graph G [10] and adding a universal vertex x increases the clique-width of $\overline{G}$ by at most 1, the clique-width of the reduced graph in [17] is at most $2\mathtt{cw}(G) + 1$. This completes the proof. □

Note that $L(1,1)$-Labeling on graphs with diameter 2 is trivially solvable because the graph power G^2 of a graph of diameter 2 is a complete graph.

The fixed-parameter tractability of $L(p,q)$-Labeling for modular-width remains open in general. On the other hand, we show that $L(1,1)$-Labeling and even $L(\mathbf{1})$-Labeling on general graphs are fixed-parameter tractable by modular-width in contrast to $L(p,q)$-Labeling.

Theorem 4. *$L(\mathbf{1})$-Labeling on general graphs is fixed-parameter tractable for modular-width.*

Proof. As mentioned in [12], $\mathtt{nd}(G) \geq \mathtt{nd}(G^k)$ holds for any graph G and any positive integer $k \geq 1$. By Proposition 2, we have $\mathtt{mw}(G) \geq \mathtt{nd}(G^2) \geq \mathtt{nd}(G^k)$ for any positive integer $k \geq 2$. Also, $L(\mathbf{1})$-Labeling on G is equivalent to Coloring on G^k. We know that Coloring is fixed-parameter tractable for neighborhood diversity [28]. Solving Coloring on G^k, one can compute $L(\mathbf{1})$-Labeling in $f(\mathtt{mw}(G))n^{O(1)}$ time. □

As the corollary of Theorem 4, we obtain an FPT-approximation algorithm for $L(\boldsymbol{p})$-Labeling with respect to modular-width.

Corollary 3. *There is a $p_{\max}$-approximation fixed-parameter algorithm for $L(\boldsymbol{p})$-Labeling on general graphs with respect to modular-width.*

Proof. For any constant c, $\lambda_{c\boldsymbol{p}} = c\lambda_{\boldsymbol{p}}$ holds. Thus, we have $\lambda_{\boldsymbol{p}} \leq \lambda_{p_{\max}\mathbf{1}} \leq p_{\max}\lambda_{\mathbf{1}}$. By Theorem 4, we obtain a $p_{\max}$-approximation fixed-parameter algorithm by modular-width. □

4.2 Subclasses of Split Graphs

Bodlaender et al. showed that $L(2,1)$-Labeling is NP-complete on split graphs of diameter at most 2 [6]. Although this implies that there unlikely exists a polynomial-time algorithm to solve $L(2,1)$-Labeling for (general) split graphs, we may expect polynomial-time algorithms by setting the target smaller classes of graphs. In this section, we show that the problem is solvable in polynomial time on block split graphs and threshold graphs. Note that the polynomial-time solvability on threshold graphs is deduced from that on co-graphs, a superclass of threshold graphs [8]. Here, we give a simple recurrence for the optimal $L(2,1)$-Labeling on threshold graphs, which provides an explicit polynomial-time algorithm.

4.2.1 Block split graphs

We first see the definitions of split graph, block graph, and block split graph. A graph $G = (V, E)$ is called a *split graph* if V can be partitioned into a clique and an independent set of G. A graph G is called a *block graph* if every biconnected component of G forms a clique. A graph G is called a *block split graph* if G is a block graph and a split graph. By the definitions of block graph and split graph, a block split graph forms a clique with *pendant* vertices, i.e., vertices with degree 1.

Theorem 5. *$L(2,1)$-Labeling on block split graphs can be solved in linear time.*

Proof. We first mention that $\lambda_{2,1}(G) \geq \max\{2(\omega(G) - 1), \Delta(G) + 1\}$ holds for a graph G, where $\omega(G)$ is the maximum clique size of G and $\Delta(G)$ is the maximum degree of G. Notice that if the bound $\Delta(G) + 1$ is tight, a vertex with degree $\Delta(G)$ (called a *major vertex*) is labeled by either 0 or $\Delta(G) + 1$; otherwise, it is easy to see that a label greater than $\Delta(G) + 1$ is necessary.

Suppose that G is a block split graph, and $\omega(G)$ and $\Delta(G)$ are respectively denoted by ω and Δ in the following. We start the argument. We consider two cases, (i) $\Delta \leq 2\omega - 4$, and (ii)

$\Delta > 2\omega - 4$. We first consider case (i). In this case, we claim that $\lambda_{2,1}$ is $2(\omega - 1)$ by giving a concrete labeling. We first label the vertices in the largest clique in G by $0, 2, \ldots, 2(\omega - 1)$. We then label pendant vertices. For a vertex in the clique, at most $\Delta - (\omega - 1)$ pendant vertices are connected. Consider a pendant vertex adjacent to a vertex labeled with $2a$, where $a \in \{0, 1, \ldots, \omega - 1\}$. Such a pendant vertex can be labeled with a label in $\{1, 3, \ldots, 2(\omega - 1) - 1\} \setminus \{2a - 1, 2a + 1\}$. Since $|\{1, 3, \ldots, 2(\omega - 1) - 1\} \setminus \{2a - 1, 2a + 1\}| \geq \omega - 3 \geq \Delta - (\omega - 1)$, all pendant vertices adjacent to a vertex with label $2a$ can be properly labeled with these labels; this matches the lower bound, and the labeling is optimal.

We next consider case (ii), and further divide it two subcases (ii-1) there are at least three major vertices, (ii-2) there are at most two major vertices. For case (ii-1), we claim that $\lambda_{2,1} = \Delta + 2$. We first see that the vertices in G can be labeled with $0, \ldots, \Delta + 2$. As case (i), we first label the vertices in the largest clique in G by $0, 2, \ldots, 2(\omega - 1)$. We then label at most $\Delta - (\omega - 1)$ pendant vertices adjacent to a common vertex with label $2a$ in the clique, where $a \in \{0, 1, \ldots, \omega - 1\}$. Such a pendant vertex can be labeled with a label in $\{2(\omega - 1) + 1, \ldots, \Delta + 2\} \cup \{1, 3, \ldots, 2(\omega - 1) - 1\} \setminus \{2a - 1, 2a + 1\}$. Since $|\{2(\omega - 1) + 1, \ldots, \Delta + 2\} \cup \{1, 3, \ldots, 2(\omega - 1) - 1\} \setminus \{2a - 1, 2a + 1\}| = \Delta + 2 - 2(\omega - 1) + (\omega - 3) = \Delta - (\omega - 1)$, all pendant vertices adjacent to a vertex with label $2a$ can be properly labeled. That is, G can be labeled by $\{0, \ldots, \Delta + 2\}$. Here, we consider the condition that there are at least three major vertices. In this case, it is impossible to label all the major vertices by either 0 or $\Delta(G) + 1$, because they are adjacent in the clique. Thus, the lower bound $\Delta(G) + 1$ on $\lambda_{2,1}$ is not tight, and actually, we have lower bound $\Delta(G) + 2$ for this case, which matches the above labeling.

We then consider (ii-2) there are at most two major vertices. If we have just one major vertex, we give label 0 to the major vertex and label the other vertices in the clique by $\{2, 4, \ldots, 2(\omega - 1)\}$. We then label at most $\Delta - (\omega - 1)$ pendant vertices adjacent to a common vertex with label $2a$ in the clique, where $a \in \{0, 1, \ldots, \omega - 1\}$. Such a pendant vertex can be labeled with a label in $\{2(\omega - 1) + 1, \ldots, \Delta + 1\} \cup \{1, 3, \ldots, 2(\omega - 1) - 1\} \setminus \{2a - 1, 2a + 1\}$. In fact, the value $|\{2(\omega - 1) + 1, \ldots, \Delta + 1\} \cup \{1, 3, \ldots, 2(\omega - 1) - 1\} \setminus \{2a - 1, 2a + 1\}|$ is $\Delta + 1 - 2(\omega - 1) + (\omega - 2) = \Delta - (\omega - 1)$ for $a = 0$, and $\Delta + 1 - 2(\omega - 1) + (\omega - 3) = \Delta - \omega$ for $a > 0$. For the $\Delta - (\omega - 1)$ pendant vertices adjacent to the major vertex (with label 0), they can be properly labeled by $\{0, \ldots, \Delta + 1\}$. Since a vertex of degree at most $\Delta - 1$ has at most $\Delta - \omega$ pendant vertices, they can be labeled again by $\{0, \ldots, \Delta + 1\}$. The lower bound $\Delta + 1$ guarantees that this labeling is optimal. If there are two major vertices, we give them labels 0 and $\Delta + 1$ and label the other vertices in the clique by $\{2, 4, \ldots, 2(\omega - 2)\}$. We then label at most $\Delta - (\omega - 1)$ pendant vertices adjacent to a common vertex with label $2a$ in the clique, where $a \in \{0, 1, \ldots, \omega - 2\}$. Such a pendant vertex can be labeled with a label in $\{2(\omega - 2) + 1, \ldots, \Delta + 1\} \cup \{1, 3, \ldots, 2(\omega - 2) - 1\} \setminus \{2a - 1, 2a + 1\}$ by a similar argument to the above. For the pendant vertices adjacent to the major vertex with label $\Delta + 1$, they can be labeled by $\{1, 3, \ldots, 2(\omega - 2) - 1\} \cup \{2(\omega - 2) + 1, \ldots, \Delta - 1\}$; since $|\{1, 3, \ldots, 2(\omega - 2) - 1\} \cup \{2(\omega - 2) + 1, \ldots, \Delta - 1\}| = \omega - 2 + (\Delta - 1 - 2(\omega - 2)) = \Delta - (\omega - 1)$, this labeling is also available and optimal. □

4.2.2 Threshold graphs

A graph G is called a *threshold graph* if there is a weight function w on V and a real number (threshold) θ such that u and v are adjacent in G if and only if $w(u) + w(v) \geq \theta$. If a graph G is a threshold graph, G is also a split graph. It is known that a threshold graph can be obtained by repeatedly adding an isolated vertex or a dominating vertex, i.e., a vertex connected to all other vertices.

Suppose that $G = (V, E)$ is a threshold graph and constructed as above. Let k be the number of dominating vertices added in the construction and x_i be the dominating vertex added at i-th step among k steps. Let I_1 be the set of the isolated vertices added before x_1, and I_i be the set of the isolated vertices added between x_{i-1} and x_i in the construction ($i \in [2, k]$). Since G is connected, x_k is added last in the construction. Note that $K = \{x_1, \ldots, x_k\}$ and $I = \bigcup_{i=1}^{k} I_i$ form a clique and an independent set, respectively. Furthermore, we have $V = K \cup I$ and $E = \binom{K}{2} \cup \bigcup_{i=1}^{k} I_i \times \{x_i, \ldots, x_k\}$. For each $i \in [k]$, let $K_i = \{x_1, \ldots, x_i\}$,$V_i = K_i \cup \bigcup_{j \in [i]} I_j$, $n_i = |V_i|$, and $G_i = G[V_i]$. Note that $n = n_k$, $G = G_k$, and every G_i is a threshold graph. If $k = 1$, then G is a star with the center x_1

and the optimal labeling can be easily obtained, so we hereafter assume $k \geq 2$. Also, without loss of generality, we assume that $I_1 \neq \emptyset$.

Lemma 1. *[8] For a graph G, the $L(p,q)$-labeling of G such that it is injection is called the $L'(p,q)$-labeling of G. We denote the minimum span among all $L'(p,q)$-labelings of G by $\lambda'_{p,q}(G)$. Also, for each disjoint graphs G and H, we define $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$ and $G + H = (V(G) \cup V(H), E(G) \cup E(H) \cup \{\{u,v\} \mid u \in V(G), v \in V(H)\})$. Then, the following holds:*

(1). $\lambda'_{2,1}(G \cup H) = \max\{\lambda'_{2,1}(G), \lambda'_{2,1}(H), |V(G)| + |V(H)| - 1\}$,

(2). $\lambda_{2,1}(G + H) = \lambda'_{2,1}(G + H) = \lambda'_{2,1}(G) + \lambda'_{2,1}(H) + 2$.

Theorem 6. *Suppose that $G_1, G_2, \ldots, G_k = G$ be a sequence of threshold graphs defined above. Then, we have $\lambda_{2,1}(G_1) = n_1$ and $\lambda_{2,1}(G_i) = \max\{n_i, \lambda_{2,1}(G_{i-1}) + 2\}$ for $i \in [2,k]$.*

Proof. For each $i \in [k]$, let $E_i = (I_i, \emptyset)$, $X_i = (\{x_i\}, \emptyset)$. Then, $G_1 = E_1 + X_1$ and $G_i = (G_{i-1} \cup I_i) + X_i$ ($i \in [2,k]$) hold. Using this representation and Lemma 1, we calculate the labeling number for each G_i. First, we get $\lambda'_{2,1}(E_i) = \max\{0, |I_i| - 1\}$ and $\lambda'_{2,1}(X_i) = 0$ immediately, so the following is obtained from Lemma 1 (2):

$$\lambda_{2,1}(G_1) = \lambda'_{2,1}(G_1) = \lambda'_{2,1}(E_1) + \lambda'_{2,1}(X_1) + 2 = \max\{2, n_1\} = n_1.$$

Next, for each $i \in [2,k]$, we get

$$\begin{aligned}\lambda'_{2,1}(G_{i-1} \cup I_i) &= \max\{\lambda'_{2,1}(G_{i-1}), \lambda'_{2,1}(I_i), n_{i-1} + |I_i| - 1\} \\ &= \max\{\lambda_{2,1}(G_{i-1}), \max\{0, |I_i| - 1\}, n_{i-1} + |I_i| - 1\} = \max\{\lambda_{2,1}(G_{i-1}), n_i - 2\}\end{aligned}$$

from Lemma 1 (1). Therefore, the following is obtained from Lemma 1 (2):

$$\lambda_{2,1}(G_i) = \lambda'_{2,1}(G_i) = \lambda'_{2,1}(G_{i-1} \cup I_i) + \lambda'_{2,1}(X_i) + 2 = \max\{\lambda_{2,1}(G_{i-1}) + 2, n_i\}.$$

□

By using this theorem, we can concretely find an optimal $L(2,1)$-labeling for each G_i as follows: We can immediately obtain an optimal $L(2,1)$-labeling of G_1 since G_1 is a star graph. For each $i \in [2,k]$, by using an optimal labeling ℓ of G_{i-1}, we construct an optimal labeling of G_i . Let $D_i = [0, \lambda_{2,1}(G_{i-1})] \setminus \ell(V(G_{i-1}))$, that is, D_i is the set of labels unused in an ℓ among $[0, \lambda_{2,1}(G_{i-1})]$. We assign labels to the $|D_i|$ vertices in I_i in D_i, and assign the other $\max\{0, |I_i| - |D_i|\}$ vertices to integers starting from $\lambda_{2,1}(G_{i-1}) + 1$. This assignment is an $L'(2,1)$-labeling of $G_{i-1} \cup I_i$ and its span is

$$\begin{aligned}\lambda_{2,1}(G_{i-1}) + \max\{0, |I_i| - |D_i|\} &= \lambda_{2,1}(G_{i-1}) + \max\{0, |I_i| - (\lambda_{2,1}(G_{i-1}) + 1 - n_{i-1})\} \\ &= \max\{\lambda_{2,1}(G_{i-1}), n_i - 2\}.\end{aligned}$$

Then, by assigning x_i to the label that is exactly 2 larger than the above span, we obtain an $L(2,1)$-labeling of G_i with span $\max\{\lambda_{2,1}(G_{i-1}) + 2, n_i\}$. This span is equal to $\lambda_{2,1}(G_i)$ since Lemma 6. Also, we can construct K and $I_1, I_2, \ldots, I_k$ in linear time.

Figure 3 shows an example of constructing an optimal labeling for a threshold graph.

5 Concluding Remarks

In this paper, we studied distance-constrained labeling problems for small diameter graphs. Our main contribution is a polynomial-time reduction from $L(\boldsymbol{p})$-Labeling on graphs of diameter at most k to Metric Path TSP. This reduction allows us to exploit various results of TSP such as exact exponential-time algorithms and approximation algorithms for solving $L(\boldsymbol{p})$-Labeling. We further investigated the parameterized complexity and the computational complexity on subclasses of small diameter graphs.

One direction of future work is to consider whether the condition in the Theorem 2 that the diameter is at most k can be generalized. Also, the parameterized complexity of $L(\boldsymbol{p})$-Labeling for modular-width remains open. Finally, it is worth investigating the computational complexity of $L(2,1)$-Labeling on block graphs, which is a superclass of trees and block split graphs.

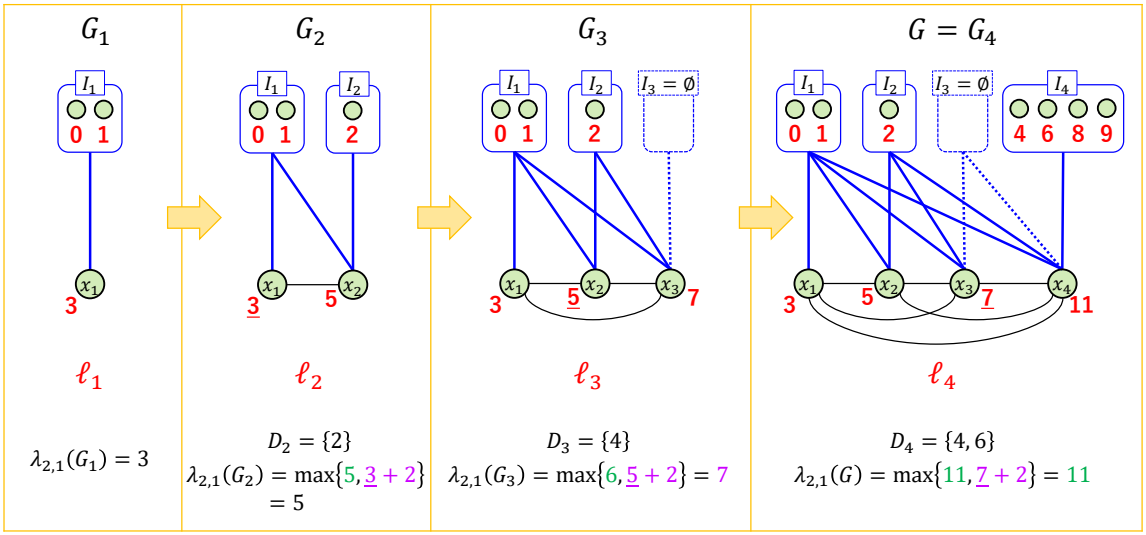

Figure 3: Construction of an optimal labeling ℓ for a threshold graph G

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